1. Section 6.2 One-to-One Functions

Definition 1.1. *A function is* **one-to-one** *if whenever you choose two different numbers* x_1 *and* x_2 *in the domain of* f*, you have* $f(x_1)$ *and* $f(x_2)$ *are also different. In other words, each value of* x *corresponds to only one* y **and** *each value of* y *corresponds to only one* x*.*

Example 1.1. *Select the one-to-one function.*

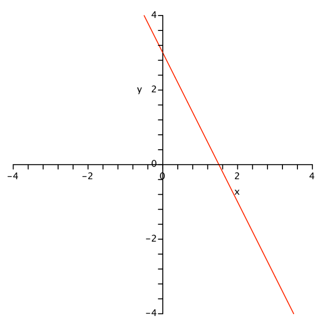
(A) $\{(2,5),(6,2),(5,3),(3,5),(4,4)\}$

(B) $\{(5,2),(2,6),(3,5),(5,3),(4,4)\}$

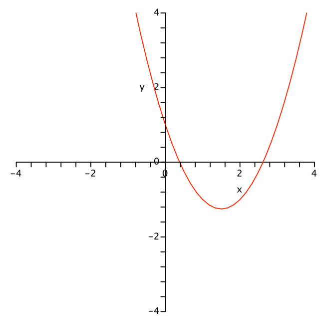
(C) $\{(5,2),(2,6),(3,5),(1,3),(4,4)\}$

Theorem 1.1 (Horizontal Line Test)**.** *A function* f *is one-to-one if and only if you cannot draw a horizontal line passing through the graph of* f *more than once.*

Example 1.2. *Select the graphs of all the one-to-one functions.*



(A)



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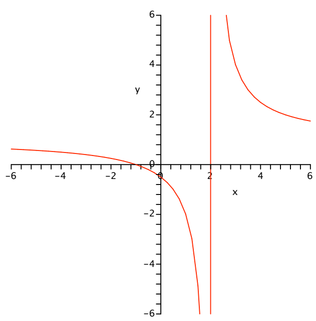

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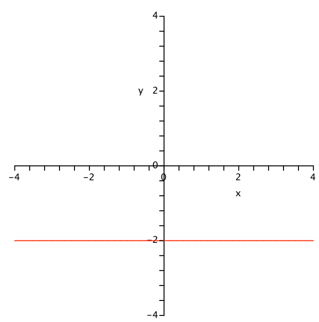

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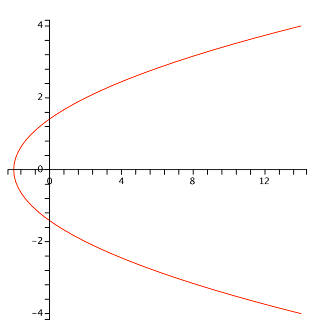
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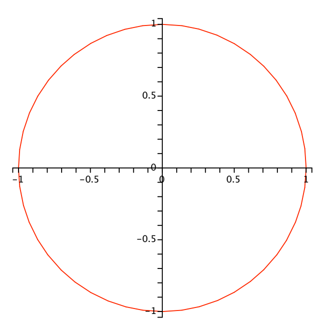

(F)

Example 1.3. *Select the formulas of all the one-to-one functions.*

(A) $f(x) = -3$

(B) $f(x) = 3x + 1$

(C) $f(x) = x^4 + 2x^3 - 5x^2 + x + 3$

(D) $f(x) = (x - 3)(x^2 - 4)$

(E) $f(x) = 2 - x^5$

2. Inverse Functions

Definition 2.1. *Let f be a one-to-one function. Then there is a function denoted f^{-1} called the* **inverse** *of f such that the domain and ranges of f and f^{-1} are interchanged and $f(a) = b$ if and only if $f^{-1}(b) = a$.*

Example 2.1. *If f is a one-to-one function and $f(3) = -5$, then what is $f^{-1}(-5)$?*

Remark 2.1. *If $f(x)$ and $g(x)$ are inverse functions, the domain of $f(x)$ is the same as the range of $g(x)$.*

3. Properties of Inverses

(1) A function g is the inverse of f (and visa versa) if and only if $(f \circ g)(x) = x$ *and* $(g \circ f)(x) = x$. Reminder: $(f \circ g)(x) = f(g(x))$.

Example 3.1. *Find $f(f^{-1}(4))$ given 4 is in the range of f.*

(2) The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

Example 3.2. *Find the domain and range for the inverse of the function $f(x) = 4x - 2$ on $[-3, 2]$. Notice $f(x)$ has a restricted domain to make $f(x)$ a line segment.*

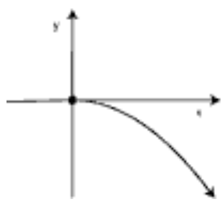
(3) (a, b) is a point on the graph of $y = f(x)$ if and only if (b, a) is a point on the graph of $y = f^{-1}(x)$.

(4) The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.

Example 3.3. *The graph of a function f is given below.*

Which of the following is the graph of $y = f^{-1}(x)$?

(A)



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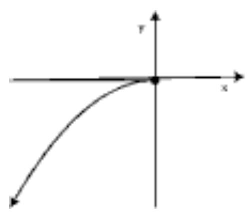

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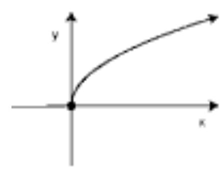

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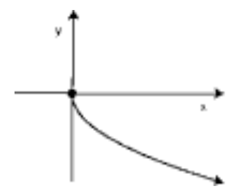

4. Finding the inverse of a function

Example 4.1. *Find the inverse of the one-to-one function* $\{(5, 2), (2, 6), (3, 5), (1, 3), (4, 4)\}$. *What is the domain and range of the inverse?*

Steps to find the inverse, $y = f^{-1}(x)$, from a one-to-one function, $y = f(x)$

(1) In $y = f(x)$, interchange the variables x and y to obtain $x = f(y)$. This equation defines the inverse function f^{-1} implicitly.
(2) Solve $x = f(y)$ for y in terms of x. Now $y = f^{-1}(x)$.
(3) Check the result by showing $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Remark 4.1. *In applications in many Applied Math courses, one must be careful of units and the variable may have significance, thus it may be preferable not to switch the variables. In this case, solve for x in the original function. For Precalculus Algebra we will always switch the variables when finding the inverse function.*

Example 4.2. *Find the formula for the inverse of the function $f(x) = \dfrac{3x-2}{1-x}$. Find the domain and range of f and f^{-1}.*

Example 4.3. *Find the formula for the inverse of the function* $f(x) = 4x - 2$ *on* $[-3, 2]$. *From Example* 3.2*, the domain of* $f(x)$ *is* $[-3, 2]$ *and the range of* $f(x)$ *is* $[-14, 6]$.

Example 4.4. *Find the formula for the inverse of the function* $f(x) = (x - 1)^2 - 2$ *on* $(-\infty, 1)$. *(The domain is restricted to make* $f(x)$ *a one-to-one function.)*

(A) $f^{-1}(x) = 1 + \sqrt{x + 2}$ *on* $(-2, \infty)$

(B) $f^{-1}(x) = 1 + \sqrt{x + 2}$ *on* $(-\infty, 1)$

(C) $f^{-1}(x) = 1 - \sqrt{x + 2}$ *on* $(-2, \infty)$

(D) $f^{-1}(x) = 1 - \sqrt{x + 2}$ *on* $(-\infty, 1)$