Hybrid asymptotic-numerical models for optimizing flexible-wing propulsion

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1 Introduction

This proposal outlines plans to create a new mathematical framework for 3D, flexible-wing propulsion that will enable high-dimensional optimization of propulsor architecture.

Currently, great efforts are being made to engineer flying and swimming vehicles that are based, not on rotary propellors, but instead on flapping wing-or-fin-like appendages. These designs bear more resemblance to bird [11, 52], insect [18, 70, 31, 78, 104, 67], and fish [76, 86, 38] locomotion than they do to classical air-or-water craft. Such biomimetic systems offer great promise for applications where maneuverability and autonomy are desired, such as surveillance, reconnaissance, salvage, and environmental sensing [86, 78]. Two such examples, a micro-air-vehicle (MAV) and an ornithopter, are shown in Figs. 1(a) and (b). Extending decades of research on the associated fluid mechanics [106, 30, 7, 22, 101, 98, 102, 97, 5, 27, 2, 46, 83, 88, 108, 29, 64, 79, 85, 107, 54, 53, 12, 80, 51, 89, 82, 84], it has been established that the use of *flexible* propulsors can drastically improve performance [27, 2, 45, 88, 62, 47, 48]. This idea has already been incorporated into the design of some existing flying/swimming devices [78, 38, 67].

To fully leverage the effects of flexibility, though, the designer must face a multitude of open questions. What is the right amount of flexibility? If the material is allowed to be distributed *nonuniformly*, where should the wing be most flexible and where should it be most rigid? What is the optimal wing/fin *geometry* for a particular application?

Questions like these can only be answered through optimization. Due to the obvious interest in designing maximally effective devices, efforts have been made to employ laboratory experiments and direct numerical simulations (DNS) in such optimization procedures [93, 2, 31, 51, 63]. For example, Tuncer and Kaya (2005) numerically simulated a wing undergoing simultaneous pitching and plunging in a 2D fluid, and optimized the amplitudes and phase-shift for thrust production and efficiency [93]. In a similar vein, Alben (2008) optimized the *uniform* elastic modulus of a 2D flapping wing for thrust production [2]. More recently, Quinn et al. (2015) combined laboratory experiments with numerical optimization techniques to maximize the propulsive efficiency of a flapping panel within a 5-dimensional space of actuation and flow parameters [63].

However, **limited by long experimental or computational run times**, these studies could only optimize over an **artificially small** parameter space of wing/fin design and

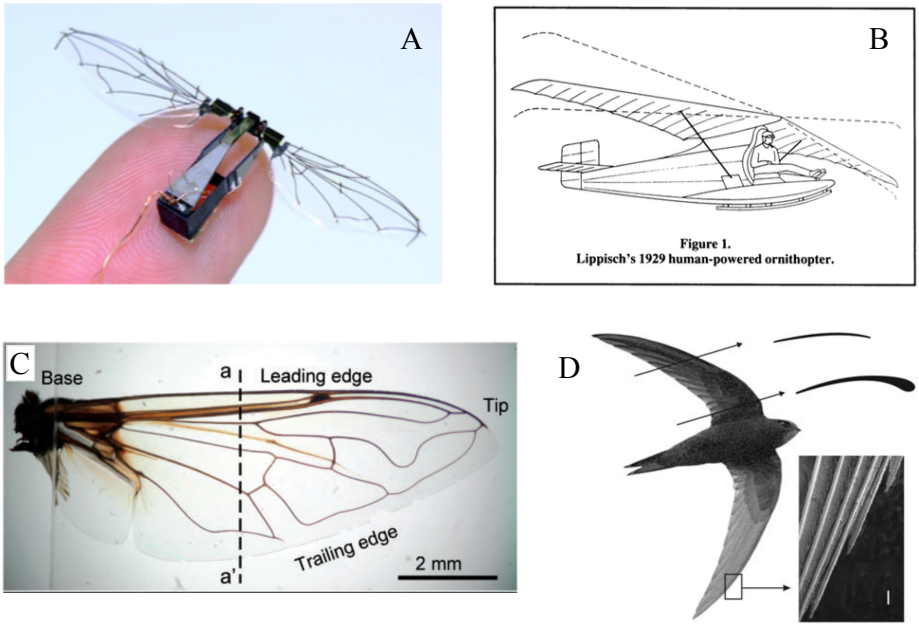

Figure 1: Many modern flying devices rely on the principles of flapping propulsion. Two examples are shown: (a) a micro-air-vehicle and (b) an ornithopter, reproduced from [78] and [11] respectively. Such devices borrow design ideas from nature, where complex material distributions and geometries prevail, as illustrated by the wings of (c) a hoverfly and (d) a swift. Images reproduced from [85] and [99] respectively.

actuation. For example, of the above studies that took into account wing flexibility, all only considered *uniform* distributions of the elastic modulus. The infinite-dimensional space of *variable* flexibility remains out of reach for such methods.

An alternative to numerical and experimental methods is the approach of *biomimicry*, where one replicates the designs found in nature and thereby relies on the optimization accomplished by eons of natural selection [86, 78, 38, 69, 104, 41, 82]. Indeed, nature presents us with wings/fins of complex material distribution and geometry, as illustrated by the examples shown in Figs. 1(c) and (d) of a hoverfly and swift wing. These architectures provide a starting point for the design of artificial devices, a few of which are shown in Figs. 1(a) and (b). However, biological systems are subject to an assortment of different constraints, and so what is optimal for life may not be optimal for a man-made device. To boot, biomimicry fails to provide a precise understanding of *why* particular designs are more effective than others.

Reduced models that capture the essential, underlying physics offer the efficiency needed for high-dimensional optimization, as well as the conceptual transparency needed to understand how particular wing designs achieve superior performance. We propose to create new hybrid asymptotic/numerical models for three-dimensional flexible-wing propulsion. We will leverage ideas from a 2D theory recently

developed by the PI [47, 48, 49]. This theory, derived from small-amplitude asymptotic analysis, couples the wing and flow dynamics through a **nonlocal operator**, allowing it to capture the **dynamic** information needed to describe flexible-wing motions [49]. The high computational efficiency enables one to rapidly scan the parameter space of variable flexibility and even perform optimization over this space. With this approach, the PI was able to determine, for the first time, certain **nonuniform** flexibility distributions that maximize propulsive performance [48].

The 3D framework will combine small-amplitude asymptotics with the idea of interacting cross-sectional domains. The latter idea is similar in spirit to lifting-line theory (LLT) [59, 60], with the important distinction that **LLT is a quasi-steady theory and thus unsuitable for describing the highly coupled dynamics between a flexible wing and fluid**. By leveraging the small-amplitude framework, we will create a theory that permits optimization over a range of *dynamic* states.

Our new theoretical framework will be used to confront a range of open scientific questions related to flexible-appendage propulsion. In particular, we will examine the role played by heterogeneous material distributions and complex geometries. Further, we will examine certain questions specific to *three-dimensional* wing design, such as: how does *anisotropic* elasticity affect propulsion [90, 10], and what is the role played by *planform* geometry? Time permitting, we will adapt the theory to handle multiple-body interactions in order to examine *cooperative* propulsors [103, 4] and to optimize their configuration and coupled actuation.

Optimizing over *all* possible aspects of propulsor architecture - including heterogeneous, anisotropic, and multi-component materials, complex cross-sectional and planform geometries, and propulsor cooperation - constitutes a grand challenge. We therefore identify a series of intermediate steps to achieve this goal, many of which will make excellent topics for PhD dissertations and undergraduate projects. In addition, we describe outreach activities designed to stimulate the interest of K-12 students in the STEM fields. The fluid dynamics of bio-inspired propulsion is a topic that naturally captures the imagination of young students and is thus particularly well-suited for these activities.

Education and training: The education and training of students in applied mathematics will be an integral part of the project, as the research will serve as the foundation for at least one PhD thesis and one undergraduate project. The students will gain broad expertise in modern applied mathematics - including the development of reduced models, the application of asymptotic/perturbation techniques, and the implementation of cutting-edge computational methods - all through a research project that cuts across the disciplines of biolocomotion and engineered propulsive systems. Students will gain experience presenting at seminars, publishing in peer-reviewed journals, and disseminating results at conferences and workshops. Of particular importance, the students will participate in conferences/workshops of an *interdisciplinary* nature, such as the APS Division of Fluid Dynamics Meeting which attracts researchers from physics, mathematics, biology, engineering, and other backgrounds. Such activities will allow the students to disseminate results to the larger scientific community, as well as expose them to the approaches/methods used in other fields. Special efforts will be made to encourage interactions, and potentially collaborations, with local researchers

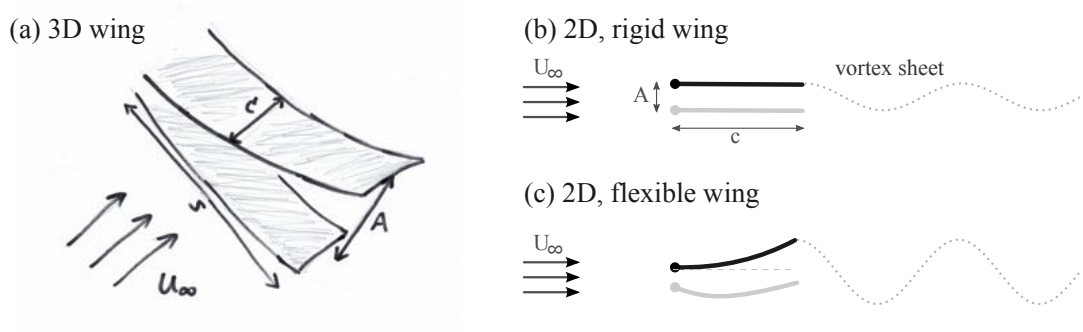

Figure 2: Diagrams. (a) A flexible wing flapping in a 3D fluid with an oncoming flow perpendicular to the flapping motion. The wing can deform along its span and chord. (b) A rigid wing and (c) a flexible wing flapping in a 2D fluid. The 2D setup can represent a cross-sectional slice of the 3D wing.

conducting laboratory experiments on hydrodynamic propulsion.

2 Theoretical framework

2.1 Background

Wings and fins achieve propulsion through a complex interaction with the surrounding fluid (usually air or water). To understand this interaction, we consider a thin wing flapping vertically in a 3D fluid, as shown in Fig. 2(a). The wing has a characteristic chord length c and span S, and it is driven with a characteristic frequency f and amplitude A in an oncoming, horizontal flow of speed U_∞. We choose the coordinate system (x, y, z) such that x aligns with the wing's chord, y with the span, and z is the vertical direction. The wing may be composed of flexible material, allowing deformations along both the chord and the span. To describe the surrounding fluid flow, we begin with the incompressible, Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = -\nabla p + \mathrm{Re}^{-1} \nabla^2 \boldsymbol{u}\,, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0\,, \tag{2}$$

where $\boldsymbol{u}$ and p are the velocity and pressure fields respectively. These equations are in dimensionless form with the Reynolds number given by $\mathrm{Re} = cU_\infty/\nu$, where ν is the kinematic viscosity of the fluid.

Motivated by the propulsive regime of birds, insects, fish, and their artificial counterparts, we will consider high Reynolds numbers (typically at least $\mathrm{Re} > 500$ and often $\mathrm{Re} > 10^4$). In this regime, Eqs. (1)–(2) can be approximated by the incompressible Euler equations ($\mathrm{Re} = \infty$) throughout the bulk of the fluid, which excludes a viscous boundary layer surrounding

the wing and a narrow wake behind it. We will develop reduced models by linearizing the Euler equations in small-flapping amplitude [106, 47, 48, 49, 58], while accounting for vortex shedding through the Kutta condition.

The small-amplitude linearization requires both $A/c \ll 1$ and $\mathrm{St} = fA/U_\infty \ll 1$. The latter ratio, known as the Strouhal number, is typically the limiting factor in applying the theory. A wide variety of flying and swimming animals have been observed to operate within the range $\mathrm{St} = 0.2 - 0.4$ [91, 87]. These Strouhal numbers, though not exceedingly small, can be reasonably modeled by small-amplitude theory [2]. In particular, birds and ornithopters operate near the bottom of the range [66], where small-amplitude theory applies most accurately. Insect flight usually occurs at higher amplitude and St [102, 78]. We would therefore not expect to describe insect flight quantitatively, although some of the qualitative findings may carry over to that case. For example, the flexibility distributions discovered by Moore (2015) [48] were found to optimize thrust *independent of driving frequency*. This suggests a robust principle that may carry over to the regime of insect flight. At the very least, the results provide a starting point to search for the optimal distributions in the higher-amplitude regime.

To determine the wing's elastic deformation, we begin with a simplified model known as Kirchhoff plate theory. This model is valid when the wing is thin and the deflections small [35, 39, 81], which dovetails nicely with the requirements for the small-amplitude flow theory. For a wing composed of isotropic elastic material, the governing equation for the vertical displacement $h(x, y, t)$ is

$$\mu \frac{\partial^2 h}{\partial t^2} + \nabla^2 \left(D \nabla^2 h \right) = q \,. \tag{3}$$

Here, D is the flexural rigidity of the wing, which may be heterogeneous $D = D(x, y)$ to allow certain parts of the wing to be more flexible than others; $\mu(x, y)$ is the wing's mass per unit area, which may also be heterogeneous. The term on the right $q(x, y, t)$ is the *hydrodynamic load*, which results from a difference in pressure between the top and bottom surfaces of the wing. We note that in the 2D case, Eq. (3) reduces to the Euler-Bernoulli beam equation for a thin wing [79, 48].

During the advanced stages of the proposed work, we will model wings with anisotropic elasticity [10], for example those whose stiffness along the chord differs from that along the span. For arbitrary anisotropic material, the linear elastic equations are significantly more complicated. The bending moments $\boldsymbol{M} = (M_{xx}, M_{yy}, M_{zz})$ are related to the curvatures $\boldsymbol{\kappa} = (h_{xx}, h_{yy}, 2h_{xy})$ linearly

$$\boldsymbol{M} = \boldsymbol{D}\boldsymbol{\kappa} \,, \tag{4}$$

where $\boldsymbol{D} = D_{ij}$ are the nine plate rigidity coefficients. The wing deformations are then governed by

$$\mu \frac{\partial^2 h}{\partial t^2} + \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2\partial_{xy} M_{xy} = q \,. \tag{5}$$

However, these equations simplify tremendously when the anisotropy depends on only two principle directions [35], for example the chord-wise and the span-wise directions. In that

case (corresponding to a cubic crystal structure) only three independent rigidity coefficients enter Eq. (4), providing a tractable starting point for analysis.

2.2 The 2D small-amplitude theory

Here, we briefly describe the small-amplitude model developed by the PI for two-dimensional wings [47, 48, 49], as many of the underlying ideas will be adopted in 3D. Importantly, the model captures the *dynamic* effects needed to describe flexible-wing/fluid interactions, which is in contrast to previous quasi-steady theories. In our model, the influence of the fluid flow enters the Euler-Bernoulli beam equation as a *nonlocal* operator that acts on the wing kinematics [24, 9, 16, 14, 75, 15, 21]. Semi-analytical solutions facilitate fast evaluation of the operator, allowing one to efficiently solve for the emergent wing motions. This efficiency enabled the PI to numerically optimize the *nonuniform* flexibility distribution of a flapping wing for maximal thrust production [48]. The mathematical and numerical aspects – for example, the discussion of the nonlocal operator, computational complexity, convergence rates, and benchmark – will be treated comprehensively in a forthcoming paper [49].

We begin by considering a thin wing flapping vertically in a 2D fluid, as diagrammed in Fig. 2(b) and (c). We consider the incompressible Euler equations, obtained by setting $\mathrm{Re} = \infty$ in Eq. (1), although we take into account the viscous production of vorticity by allowing the wing to shed a vortex sheet from its trailing edge [74, 42]. Taking the driving amplitude and the associated vertical velocities to be small, i.e. $A/c \ll 1$ and $fA/U_\infty \ll 1$, allows the Euler equations to be linearized [106], giving

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\boldsymbol{u} = \nabla\phi\,, \tag{6}$$

$$\nabla\cdot\boldsymbol{u} = 0\,. \tag{7}$$

Here, $U = 2U_\infty/(cf)$ is the dimensionless free-stream velocity, $\boldsymbol{u} = (U+u, v)$ is the velocity field, and $\phi = 4(p_\infty - p)/(\rho c^2 f^2)$ is the so-called Prandtl acceleration potential [106], which is simply a normalized, negative pressure.

Taking the divergence of Eq. (6) and using linearity shows that ϕ is a harmonic function

$$\nabla^2\phi = 0 \tag{8}$$

The flow is subject to no-penetration boundary conditions on the wing

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)h = v \qquad \text{for } z = \pm 0, -1 < x < 1 \tag{9}$$

where $h(x,t)$ is the wing's vertical displacement. Meanwhile, vorticity is shed according to the Kutta condition, $|\boldsymbol{u}| < \infty$, at the trailing edge.

In the case of *prescribed* time-harmonic kinematics $h(x,t) = e^{2\pi jt}\eta(x)$, this system admits exact solutions for the pressure field, ϕ. The solutions are found by performing a Chebyshev expansion of $\eta(x)$, using Eq. (9) to obtain boundary conditions on a harmonic conjugate

function ψ, inserting a multipole expansion for the analytical function $F(z) = \phi + i\psi$, and solving for the coefficients in the expansion after conformally mapping to a suitable domain [106, 47].

With ϕ determined, it is possible to calculate the *hydrodynamic load* on the wing, $q(x,t) = \phi(x,0^+,t) - \phi(x,0^-,t)$, i.e. the pressure difference between the top and bottom surfaces. The resulting load is harmonic in time and given by

$$q(x,t) = e^{2\pi jt} Q(x)\,, \tag{10}$$

$$Q(x) = a_0 \sqrt{\frac{1-x}{1+x}} + 2\sum_{k=1}^{\infty} a_k \sin k\theta\,. \tag{11}$$

The coefficients a_k in the series depend linearly on $\eta(x)$ [106], and thus the hydrodynamic load can be thought of as a *linear, nonlocal operator* that acts on the kinematics, i.e. $Q = Q[\eta(x)]$. Due to the relationship between the harmonic conjugate functions ϕ and ψ, the nonlocal operator involves a Hilbert transform [92] and, furthermore, can be understood in terms of Riemann-Hilbert analysis [1, 28, 57, 50]. Given the wing kinematics, $\eta(x_n)$, at N Chebyshev collocation points, the PI has devised a method to efficiently compute the action of the operator $Q[\eta(x_n)]$ with $O(N \log N)$ operations [49]. The method relies on first computing $\psi(x_n)$ through Chebyshev collocation and then computing $\phi(x_n)$ through a (specialized form of the) Hilbert transform.

Now consider the situation of a *flexible* wing, where a driving motion is prescribed (for example at the leading edge), but the wing's *deformation* depends on the coupling between the fluid and elastic forces. In that case, the hydrodynamic load couples to Eq. (3) and, combined with the time-harmonic assumption, gives the equation for $\eta(x)$,

$$\mathcal{D}^2\left(S(x)\mathcal{D}^2\eta\right) - 3\sigma^2 R(x)\eta = \frac{3\sigma^2}{8\pi^2} Q[\eta](x)\,, \tag{12}$$

where $\mathcal{D} = d/dx$. This equation is in dimensionless form and we have introduced three new parameters: $\sigma = 2\pi/U$ is the reduced driving frequency, $S(x)$ is the dimensionless stiffness, and $R(x)$ is the ratio of solid-to-fluid inertia. Since we are interested in wings of *nonuniform* material composition, we have allowed S and R to vary with distance along the wing, x. **Importantly, no restrictions are placed on the reduced frequency, σ, which allows the theory to capture a range of *dynamic* behaviors.**

Equation (12) is a differential equation with a nonlocal term $Q[\eta]$, and the PI has devised a method to numerically solve this equation with $O(N\, logN)$ operations [47, 49]. The method is based on the ability to apply $Q[\eta]$ forward, and it uses Chebyshev spectral collocation to treat the differential operators in Eq. (12) [13, 44, 23]. The system is solved iteratively using GMRES [73, 71, 72], after preconditioning the *continuous* form of Eq. (12) (essentially converting it to an integral equation) [20]. Preconditioning not only remedies the poor condition of the fourth-order differential operator [19, 33, 3, 32, 43], but also allows analytical removal of the singular term, $\sqrt{(1-x)/(1+x)}$, present in Eq. (11).

The efficiency of the algorithm renders high-dimensional optimization feasible, and this task was carried out by Moore (2015) for wings of arbitrary flexibility distributions [48].

The main finding was that focusing flexibility near the wing's leading edge enhances thrust production significantly. Placing a torsional spring at the leading edge focuses all of the flexibility there, and this arrangement was found to *globally* optimize thrust production, independent of the wing's mass ratio and the driving regime.

2.3 The proposed 3D framework

A primary objective of this proposal is to create an analogous 3D model for coupled wing-fluid dynamics. We envision the 3D model to be based on the idea of 2D cross-sections that interact through a downwash and, possibly, a span-wise flow. This idea is similar in spirit to lifting-line theory (LLT), with an important distinction: LLT is a *quasi-steady* theory, whereas we will exploit the small-amplitude framework to develop a *dynamic* theory. We will, nonetheless, borrow a few key ideas from LLT and so we provide a brief sketch of the theory here.

2.3.1 Salient ideas from lifting-line theory

Lifting-line theory was originally created to determine the the steady aerodynamic forces produced by wings of *finite* span [61] and has since been used in a range of applications [8, 60, 59]. LLT relies on the idea of decomposing a 3D flow into a series of 2D cross-sections, with appropriate communication between the sections. For this reason, it is sometimes regarded as a quasi-3D theory, although Nierop et al. (2005) demonstrated its ability to capture subtle, 3D geometric effects related to hydrodynamic performance of whale flippers [96]. This study gives us confidence that the concept of interacting cross-sectional domains can adequately describe the role of planform geometry in flapping propulsion.

Traditional lifting-line theory supposes that the flow around each cross section is approximately 2D potential flow with vorticity shed according to the Kutta condition. As such, the circulation K in each domain depends on the cross-sectional geometry and angle of attack α. Taking the example of an elliptic wing with a cross-sectional Joukowski-profile [96], this dependence is given by

$$K \sim -\pi U \left(c + \frac{4}{3\sqrt{3}} b \right) \alpha^e \tag{13}$$

where α^e is the effective angle of attack, which is unknown initially. The wing chord c and thickness b may vary along the span y, thus creating a circulation that also varies with span, i.e. $K = K(y)$. The circulation creates a downwash $w(y)$, which in turn alters the effective angle of attack through

$$\alpha^e = \alpha - w/U \tag{14}$$

Often, the wing itself is represented by a single vortex line, typically lying along the quarter chord. The circulation and the downwash are related through

$$w(y) = \frac{1}{4\pi} \int \frac{dK}{dy'} \frac{1}{y - y'} dy' \,, \tag{15}$$

which, in light of Eqs. (13) and (14), can be viewed as an integral equation for either $K(y)$ or $w(y)$. It is through this equation that different cross sections communicate to create a 3D flow [59, 60, 96]. Typically, the above integral equation is solved numerically [92, 68]. We comment that, for a real wing, a span-wise flow is also present but is not captured by LLT. However, it has been shown that the span-wise flow does not contribute to lift or thrust, and thus can reasonably be neglected [59].

2.3.2 The 3D small-amplitude theory

We aim to combine the idea of communicating cross-sectional domains with the small-amplitude theory in order to create a *dynamic* 3D-flow model capable of describing flexible-wing interactions. We remark that in the small-amplitude theory, the variable of focus is *pressure field*, as opposed to say the velocity or vorticity field. This is a great advantage of the theory, as it is pressure that couples directly to the elastic balance and thus creates wing deformation. The flow and vorticity fields are auxiliary variables that typically do not need to be computed. However, creating a 3D model (at least in the way envisioned here) requires the inclusion of a downwash flow that is created by circulation, and therefore the vorticity field will be required.

The steps required for creating the 3D small-amplitude framework are as follows:

1. Modify the small-amplitude theory to allow for a *time-varying* angle of attack $\alpha(t)$.

2. From the resulting Prandtl acceleration potential, ϕ, determine the vorticity field and the subsequent *time-dependent* circulation $K(y,t)$ in each cross-sectional domain.

3. Use Eq. (14) to link $\alpha(t)$ and a downwash $w(y,t)$, and then Eq. (15) to link $w(y,t)$ and $K(y,t)$ (this step the same as classical lifting-line theory).

4. Solve the resulting integral equation to determine the 3D flow that is consistent with all of the above relationships.

As we have emphasized, an important difference with classical LLT is that, here, the circulation, downwash, and attack angles must all *vary in time*. It is for this reason that we appeal to the small-amplitude analysis rather than using the quasi-steady relationship from Eq. (13).

Step 1 above probably represents the greatest challenge. Previous attempts have been made to include an angle of attack in the small-amplitude theory, but were not met with success [36]. In that study, the motivation was to develop a kinetic theory capable of describing multi-body interactions. The main obstacle was that, with an arbitrary angle of attack, the Prandtl acceleration potential is no longer harmonic, but instead satisfies a Poisson equation which was not amenable to exact solution. Consequently, the 'cell problem' could not be solved [36]. In our case, however, the resulting downwash will be the same order of magnitude as the driving amplitude, i.e. asymptotically small, which potentially offers a saving grace. We will exploit this scaling relationship to extend the theory for small angles of attack and thus make it possible to develop the desired 3D model.

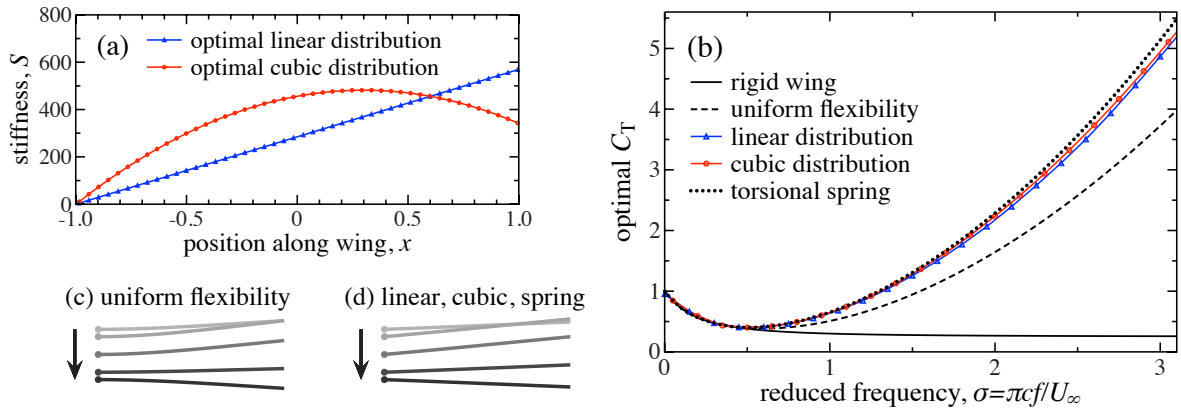

Figure 3: Optimizing a wing's elastic distribution in 2D. (a) The optimal arrangements for a linear (blue) and cubic (red) distribution of stiffness. Both show flexibility concentrated at the leading edge of the wing. (b) The thrust coefficient produced by the optimal uniform (dashed), linear (blue), and cubic (red) distributions, as well as for the optimal torsional spring (dotted). For comparison, we also show C_T produced by a rigid wing (black). (c) and (d), the emergent wing kinematics for the (c) wing of optimal uniform flexibility and the (d) optimal linear distribution. Figure adapted from Moore (2015) [48].

3 Optimizing wing architecture

Our primary objective is to use these new models to optimize the architecture of flapping wings for maximal performance. This task includes optimizing the distributions of elastic modulus and mass density, as well as the geometry of the wing. Below, we discuss relevant results that have already been established in 2D and proposed directions of study in both 2D and 3D. Throughout, the main metric of performance will be thrust, motivated by its importance in bird, insect, and fish propulsion [6, 18, 77, 17, 94]. Other metrics, such as propulsive efficiency [87, 95], may also be considered in the future.

3.1 Optimization results in 2D

Natural selection has created propulsors of complex geometries and material distributions, as seen in Fig. 1(c) and (d). In the example of a hoverfly wing, Fig. 1(c), a network of tubular veins lend structural integrity to the wing, while allowing a certain amount of flexibility that is distributed nonuniformly [85]. Recent direct numerical simulations have investigated the effects of such nonuniform elastic distributions [80]. However, limited by computational cost, these simulations could only examine a few different distributions, and optimization was not attempted.

In contrast, the efficiency of the small-amplitude framework renders optimization feasible, and this task has already been carried out by the PI in two dimensions [48]. In that study, the elastic modulus was allowed to vary along the chord of the wing. The main finding

was that focusing flexibility near the wing's *leading edge* improves thrust significantly. To arrive at this conclusion, the study considered various forms of elastic distributions - first a constant modulus, then linear distributions, then quadratic, cubic, etc. - each of which better approximates an arbitrary distribution. For each family, the elastic distribution was numerically optimized [37, 109, 55, 105] for thrust production.

Figure 3(a) shows the linear and cubic distributions that result from the optimization at a particular driving frequency. Both of these optimal wings are most flexible at the leading edge and more rigid further back. Figure 3(b) shows the thrust coefficient C_{T} achieved by the optimal wings over a range of driving frequencies σ (the optimization is performed at each frequency individually). For comparison, the black curve shows C_{T} produced by a rigid wing. The optimal wings, which have their flexibility focused at the leading edge, each produce significantly more thrust than the rigid wing. Furthermore, the optimal cubic distribution yields only a slight improvement over the linear distribution, suggesting convergence in distribution space. Taking the trend of focused flexibility to the extreme produces the case of a *torsional spring* affixed to the leading edge of an otherwise rigid wing. Shown by the dotted curve in Fig. 3(b), this arrangement produces more thrust than any other. In fact, the torsional-spring setup was found to globally optimize thrust, *independent of the wing's mass ratio and its driving frequency* [48].

3.2 Still-open questions in 2D

Several important questions remain open in the modeling of two-dimensional wings. A few of these questions are:

1. What is the *mass-density* distribution that optimizes thrust?
2. What is the *cross-sectional geometry* that optimizes thrust?
3. How do these results change when other performance metrics, such as *propulsive efficiency*, are targeted?

These tasks, since they involve modifications of the existing 2D framework, would be particularly valuable for undergraduate or beginning graduate students to gain exposure to the field.

Furthermore, each of these questions has important implications for understanding biological propulsion and creating artificial designs. Regarding question (1) for example, some species of insects have a concentrated mass, known as a pterostigma, near the outer edge of the wing [31]. It is hypothesized that this additional inertia serves as a passive mechanism for increasing aerodynamic forces. This idea that could be tested by our small-amplitude model. In fact, it would be very interesting to see if such a mechanism emerges natural from the processes of optimizing mass distribution.

Regarding question (2), nature presents us with complex and fascinating wing shapes, as shown in Fig. 1(a) and (b). In particular, the fast-flying bird species called swifts exhibit wings with a swept-back shape, which is thought to contribute to their superior aerodynamic

performance [99]. Rather than analyze individual wing shapes observed in nature, we propose to determine which geometries emerge from an optimization procedure. To make this goal feasible, we will first use the 2D model to examine the effect of cross-sectional geometry in isolation. This would require a simple modification of the small-amplitude theory by introducing different conformal maps to account for different geometries (e.g. the Joukowsky transform [74]), making an excellent challenge for an advanced undergraduate student. A recent study combined laboratory experiments with a genetic algorithm in order to determine the optimal cross-sectional shape [65], and this study could be used for comparison.

3.3 Optimizing wing architecture in 3D

Our primary objective is to use the newly developed theoretical framework to optimize three-dimensional wing architecture. In particular, we will consider wings composed of heterogeneous, anisotropic, and multi-component materials, having complex cross-sectional and planform geometries. Clearly, such considerations greatly increase the parameter space that is needed and the complexity that is possible. To make this challenge accessible, we have broken it down into a series of intermediate steps.

Regarding the distribution of **wing elasticity**, we propose to

1. Optimize the isotropic flexibility and mass distributions of a rectangular wing of given dimensions and actuation.

2. Consider the more general case of *anisotropic* flexibility. Determine the optimal *uniform but anisotropic* elastic modulus for a wing of given dimensions and actuation. Compare the performance to the wing of optimized uniform, isotropic modulus in order to assess the role of anisotropy in the simplest case.

3. Consider the most general case of *nonuniform and anisotropic* flexibility, and optimize the distribution of the elastic modulus.

Regarding **wing geometry**, we propose:

1. For a rigid wing with rectangular planform geometry, optimize its cross-sectional geometry. This task is closely related to optimizing the geometry in 2D, but there may be subtle effects introduced by the wing's finite span.

2. For a rigid wing of fixed cross section and actuation, optimize the planform geometry.

3. For a rigid wing, simultaneously optimize the cross-sectional and planform geometry as they are allowed to vary independently.

After these steps are completed, the grand challenge will be to simultaneously optimize a wing's overall geometry (cross-section and planform), as well as its flexibility (isotropic or anisotropic) and mass distributions for maximal performance.

Interestingly, previous results suggest that we may encounter a fundamental difference in how span-wise and chord-wise deformations affect performance. Measurements on the

span-wise deformations of biological wings and fins point to an advantage in concentrating flexibility near the wing/fin tip [40], which intuitively contrasts with the idea of focusing flexibility near the leading-edge of a 2D wing [48]. This distinction, if real, could be born out by our optimization tests of anisotropic wings.

4 Broader impacts of the proposed work

The results of this project will help guide future designs of biomimetic flying/swimming vehicles. Furthermore, contacts with faculty in FSU's Mechanical Engineering department, specifically the group of Professor Kunihiko Taira, will accelerate these results reaching real-world applications. The newly developed tools will allow us to examine important scientific questions unique to three dimensions, such as how material anisotropy and planform geometry influence propulsion.

The above scientific impacts have been addressed throughout this proposal, and so here we emphasize how this project will impact future generations of mathematical scientists through educational and outreach activities. The fluid dynamics of bio-like propulsion is a topic that naturally captures the imagination of young students and thus serves as a wonderful opportunity in this regard. The PI is already actively involved in graduate and undergraduate research, as well as outreach to K-12 students. Below, we describe a range of new educational activities related to the proposed research.

4.1 Graduate seminar 'Propulsion in fluids: How does a bird fly?'

The PI will create a special graduate seminar course with a simple goal in mind: for the students to acquire the assortment of mathematical tools and physical reasoning needed to explain how a bird achieves flight.

The course will build the necessary theory from the ground up, beginning with the surrounding high-Reynolds-number flows and covering: (A) the powerful simplification offered by potential-flow theory; (B) the thrust-generating mechanism of vortex shedding and its origin from viscous boundary layers; (C) the creation of circulation and lift as aided by wing camber and angle-of-attack.

Next, to understand the articulate motions of natural wings and fins, exposure to elastic mechanics is required. Emphasis will be placed on linear elastic theory and the simplifications offered by Euler-Bernoulli beam theory and Kirchhoff plate theory – two classical examples of model reduction from asymptotic analysis of PDEs.

The course will culminate with a description of how elastic wings/fins *interact* with the surrounding flows through nonlinear coupling of the boundary conditions. We will examine the range of possible behaviors from the perspective of a high-dimensional dynamical system. By drawing together ideas from PDE-theory, asymptotic analysis, and dynamical systems, the course will be framed to appeal generally to graduate students in our program (not just those working in fluid dynamics) and will deepen their exposure to applied mathematics.

Cross-disdiplinary outreach: Even more broadly, I expect the course to appeal to graduate students in other STEM fields. I already have experience developing this type of cross-disciplinary seminar: in Fall 2014, I taught a course on 'Modeling fluid flows in geophysics', which was received well, not just by mathematics students, but also by students in the neighboring Geophysical Fluid Dynamics Institute (many of whom I have continued contact with and serve on their PhD committees). I anticipate the proposed course to attract students from FSU's Mechanical Engineering program, due to my connections with faculty and graduate students in that department (I am serving as a PhD committee member for one such student). This would be an important step in fostering collaboration with that department.

4.2 Integration into the standard FSU graduate curriculum

After teaching the new seminar course, I plan to integrate the material into our standard graduate curriculum. While the fluid-mechanics component complements an existing course we have, elastic mechanics is not covered elsewhere in our program. Within the Applied and Computation Mathematics (ACM) program at FSU, our first-year courses are well developed, but, in discussions with our Chair and ACM director, we have identified a few **critical gaps** in our second-and-third-year courses. Material from the seminar course could close two of these gaps: continuum mechanics and mathematical modeling. In the coming years, I have been asked to create a 2nd-year mathematical-modeling course (which could easily include continuum mechanics as a topic). The material from the proposed seminar will be instrumental in developing this new course.

4.3 Undergraduate research

Flapping propulsion makes a rich and fascinating topic for undergraduate research, while also being approachable by the methods and tools acquired at that stage. As mentioned above, there are a number of open questions in this proposal appropriate for undergraduate research, beginning with using the existing 2D framework to explore the role of mass distribution and cross-sectional geometry in propulsion.

At FSU, there are (at least) two excellent avenues to attract interested undergraduate students: the Undergraduate Research Opportunity Program (UROP) and the IDEA grant. The UROP is a program to engage underclassmen in academic research, while the IDEA grant is a competitive program, requiring (typically advanced) students to write a proposal in which they identify a research advisor. Both programs make special efforts to recruit from underrepresented groups. I already have experience using these programs to recruit interested students as I am currently serving as the advisor for IDEA grant winner Tyler Bolles (whose proposal was ranked #1 overall out of the 110 applicants).

I will use a combination of funds from the above two resources and from the current proposal to engage undergraduates in research. Since the UROP program is well-known at FSU, it is the best way to recruit young, interested students. I plan to recruit one such student who will be supported by UROP for the first summer of his/her research, and should

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Mathematics For Optimal Wing Design

Kevin Knudson
CONTRIBUTOR

Evolution has done a pretty good job of producing efficient wings in birds and insects. In particular, natural wings tend to be quite flexible and this allows animals to generate lift and thrust with minimal energy. Here's a drone beetle in a wind tunnel captured with an ultra high-speed camera. Note how much the wings twist and flex as the beetle flaps them. In particular, the wing flexes along its front edge while it moves up and down.

Figure 4: A general-public article featured in the Forbes Science section [34] that discusses the results of Moore (2015) [48].

the student wish to continue, I will fund subsequent years using the current proposal. I will also encourage the student to apply for the IDEA grant in the final year(s) of research, as this is a prestigious award at FSU.

4.4 Outreach for K-12 and the general public

The question of how birds, insects, and fish achieve propulsion is one that naturally captures the public's imagination, as illustrated by a number of popular articles that have been written about the PI's research [34, 56, 26, 25] (see the clip in Fig. 4). I have already seized this opportunity to engage K-12 students in outreach activities. In particular, my research was the focus of an educational video created by **CPALMS** - the State of Florida's official source for standards information and course descriptions for K-12 education. The video, which features an interview with me and animations of my research, is used to reinforce concepts from mathematics courses (grade levels 7-12), and to encourage students to consider a career in the STEM fields [100]. Additionally, these videos are frequently used for the continuing education of K-12 *teachers*, providing them a broader perspective of mathematics that could be integrated into their teaching. I plan to continue this collaboration with the CPALMS foundation to create a series of educational videos for both students and teachers.

5 Results from Prior and Current NSF Support: N/A

References

[1] M.J. Ablowitz and A.S. Fokas. *Complex Variables: Introduction and Applications*. Cambridge University Press, 2003.

[2] S. Alben. Optimal flexibility of a flapping appendage in an inviscid fluid. *J. Fluid Mech.*, 614:355, 2008.

[3] D. Arnold, R. Falk, and R. Winther. Preconditioning in H(div) and applications. *Math. Comput.*, 66(219):957–984, 1997.

[4] A.D. Becker, H. Masoud, J.W. Newbolt, M. Shelley, and L. Ristroph. Hydrodynamic schooling of flapping swimmers. *Nature communications*, 6, 2015.

[5] A.J. Bergou, S. Xu, and Z. Wang. Passive wing pitch reversal in insect flight. *J. Fluid Mech.*, 591:321–337, 2007.

[6] P. Burgers and L.M. Chiappe. The wing of archaeopteryx as a primary thrust generator. *Nature*, 399(6731):60–62, 1999.

[7] S. Childress. *Mechanics of swimming and flying*, volume 2. Cambridge University Press, 1981.

[8] L.P. Cook and J.D. Cole. Lifting line theory for transonic flow. *SIAM J. Appl. Math.*, 35(2):209–228, 1978.

[9] J. Coville. On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators. *J. Differ. Equations*, 249(11):2921–2953, 2010.

[10] S.C. Cowin. Properties of the anisotropic elasticity tensor. *The Quarterly Journal of Mechanics and Applied Mathematics*, 42(2):249–266, 1989.

[11] J.D. DeLaurier. An ornithopter wing design. *Can. Aeronaut. Space J.*, 40(1):10–18, 1994.

[12] P.A. Dewey, B.M. Boschitsch, K.W. Moored, H.A. Stone, and A.J. Smits. Scaling laws for the thrust production of flexible pitching panels. *J. Fluid Mech.*, 732:29–46, 2013.

[13] W.S. Don and A. Solomonoff. Accuracy and speed in computing the Chebyshev collocation derivative. *SIAM J. Sci. Comput.*, 16(6):1253–1268, 1995.

[14] Q. Du, M. Gunzburger, R.B. Lehoucq, and K. Zhou. Analysis and approximation of nonlocal diffusion problems with volume constraints. *SIAM Rev.*, 54(4):667–696, 2012.

[15] Q. Du, M. Gunzburger, R.B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. *Math. Mod. Meth. Appl. S.*, 23(03):493–540, 2013.

[16] Qiang Du and Kun Zhou. Mathematical analysis for the peridynamic nonlocal continuum theory. *ESAIM-Math Model. Num.*, 45(2):217–234, 2011.

[17] R. Dudley. Mechanisms and implications of animal flight maneuverability. *Integr. Comp. Biol.*, 42(1):135–140, 2002.

[18] C.P. Ellington. The novel aerodynamics of insect flight: applications to micro-air vehicles. *J. Exp. Biol.*, 202(23):3439–3448, 1999.

[19] H.C. Elman and M.H. Schultz. Preconditioning by fast direct methods for nonself-adjoint nonseparable elliptic equations. *SIAM J. Numer. Anal.*, 23(1):44–57, 1986.

[20] David J Evans. *Preconditioned Iterative Methods*. Gordon and Breach Science Publishers, Inc., 1994.

[21] M. Felsinger and M. Kassmann. Local regularity for parabolic nonlocal operators. *Commun. Partial Diff. Eq.*, 38(9):1539–1573, 2013.

[22] F.E. Fish. Power output and propulsive efficiency of swimming bottlenose dolphins (tursiops truncatus). *J. Exp. Biol.*, 185(1):179–193, 1993.

[23] A. Gil, J. Segura, and N.M. Temme. *Numerical methods for special functions*. SIAM, 2007.

[24] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Modeling & Simulation*, 7(3):1005–1028, 2008.

[25] Kathleen Haughney. Crunching numbers: Math equations help build optimal bird wing. *Florida State 24/7: the news site of Florida State University*, 2015. `http://news.fsu.edu/More-FSU-News/Crunching-numbers-Math-equations-help-build-optimal-bird-wing`.

[26] Kathleen Haughney. Math equations help build optimal bird wing. *Tallahassee Democrat*, 2015. `http://www.tallahassee.com/story/life/family/2015/09/29/math-equations-help-build-optimal-bird-wing/73053308/`.

[27] S. Heathcote and I. Gursul. Flexible flapping airfoil propulsion at low reynolds numbers. *AIAA J.*, 45(5):1066–1079, 2007.

[28] A. Its. The Riemann–Hilbert problem and integrable systems. *Notices of the AMS*, 50(11):1389–1400, 2003.

[29] C. K. Kang, H. Aono, C. E. S. Cesnik, and W. Shyy. Effects of flexibility on the aerodynamic performance of flapping wings. *J. Fluid Mech.*, 689(1):32–74, 2011.

[30] J. Katz and D. Weihs. Hydrodynamic propulsion by large amplitude oscillation of an airfoil with chordwise flexibility. *J. Fluid Mech.*, 88(03):485–497, 1978.

[31] Z. Khan, K. Steelman, and S. Agrawal. Development of insect thorax based flapping mechanism. In *IEEE Int. Conf. Robot.*, pages 3651–3656. IEEE, 2009.

[32] Sang Dong Kim and Seymour V Parter. Preconditioning Chebyshev spectral collocation by finite-difference operators. *SIAM J. Numer. Anal.*, 34(3):939–958, 1997.

[33] S.D. Kim and S.V. Parter. Preconditioning Chebyshev spectral collocation method for elliptic partial differential equations. *SIAM J. Numer. Anal.*, 33(6):2375–2400, 1996.

[34] Kevin Knudson. Mathematics for optimal wing design. *Forbes Science Section*, 2015. `http://www.forbes.com/sites/kevinknudson/2015/10/02/mathematics-for-optimal-wing-design/`.

[35] L.D. Landau and E.M. Lifshitz. *Elasticity theory*. Pergamon Press, 1975.

[36] L. Li. *Asymptotic and numerical analysis of fluid-structure interactions at different Reynolds numbers*. PhD thesis, The University of Wisconsin-Madison, 2015.

[37] D.C. Liu and J. Nocedal. On the limited memory BFGS method for large scale optimization. *Math. Program.*, 45(1-3):503–528, 1989.

[38] J. Liu and H. Hu. Biological inspiration: from carangiform fish to multi-joint robotic fish. *Journal of Bionic Engineering*, 7(1):35–48, 2010.

[39] A.E.H. Love. *A treatise on the mathematical theory of elasticity*, volume 1. Cambridge University Press, 2013.

[40] K.N. Lucas, N. Johnson, W.T. Beaulieu, E. Cathcart, G. Tirrell, S.P. Colin, B.J. Gemmell, J.O. Dabiri, and J.H. Costello. Bending rules for animal propulsion. *Nature Commun.*, 5, 2014.

[41] K.N. Lucas, P.J.M. Thornycroft, B.J. Gemmell, S.P. Colin, J.H. Costello, and G.V. Lauder. Effects of non-uniform stiffness on the swimming performance of a passively-flexing, fish-like foil model. *Bioinspiration & biomimetics*, 10(5):056019, 2015.

[42] Andrew Majda. Vorticity and the mathematical theory of incompressible fluid flow. *Communications on Pure and Applied Mathematics*, 39(S1), 1986.

[43] K.A. Mardal and R. Winther. Preconditioning discretizations of systems of partial differential equations. *Numer. Linear Algebr.*, 18(1):1–40, 2011.

[44] J.C. Mason and D.C. Handscomb. *Chebyshev polynomials*. CRC Press, 2002.

[45] H. Masoud and A. Alexeev. Resonance of flexible flapping wings at low reynolds number. *Phys. Rev. E*, 81(5):056304, 2010.

[46] S. Michelin and S.G.L. Smith. Resonance and propulsion performance of a heaving flexible wing. *Phys. Fluids*, 21:071902, 2009.

[47] M.N.J. Moore. Analytical results on the role of flexibility in flapping propulsion. *J. Fluid Mech.*, 757:599–612, 2014.

[48] M.N.J. Moore. Torsional spring is the optimal flexibility arrangement for thrust production of a flapping wing. *Phys. Fluids*, 27(9):091701, 2015.

[49] M.N.J. Moore. A fast Chebyshev method for simulating flexible-wing propulsion. *J. Comput. Phys.*, 345:792–817, 2017.

[50] M.N.J. Moore. Riemann-Hilbert problems for the shapes formed by bodies dissolving, melting, and eroding in fluid flows. *Comm. Pure Appl. Math.*, 2017.

[51] K.W. Moored, P.A. Dewey, B.M. Boschitsch, A.J. Smits, and H. Haj-Hariri. Linear instability mechanisms leading to optimally efficient locomotion with flexible propulsors. *Phys. Fluids*, 26(4):041905, 2014.

[52] T.J. Mueller and J.D. DeLaurier. Aerodynamics of small vehicles. *Annu. Rev. Fluid Mech.*, 35(1):89–111, 2003.

[53] T. Nakata and H. Liu. Aerodynamic performance of a hovering hawkmoth with flexible wings: a computational approach. *Proceedings of the Royal Society of London B: Biological Sciences*, 279(1729):722–731, 2012.

[54] T. Nakata and H. Liu. A fluid–structure interaction model of insect flight with flexible wings. *J. Comput. Phys.*, 231(4):1822–1847, 2012.

[55] S.G. Nash and J. Nocedal. A numerical study of the limited memory BFGS method and the truncated-Newton method for large scale optimization. *SIAM J. Optimiz.*, 1(3):358–372, 1991.

[56] Bob Novella. Ornithopters: Making maximally-efficient man-made wings. *The Skeptics' Guide to the Universe: Science News*, 2015. `http://www.theskepticsguide.org/ornithopters-making-maximally-efficient-man-made-wings`.

[57] S. Olver and T. Trogdon. Nonlinear steepest descent and numerical solution of Riemann–Hilbert problems. *Comm. Pure Appl. Math.*, 67(8):1353–1389, 2014.

[58] F. Paraz, L. Schouveiler, and C. Eloy. Thrust generation by a heaving flexible foil: Resonance, nonlinearities, and optimality. *Phys. Fluids*, 28(1):011903, 2016.

[59] W.F. Phillips and D.O. Snyder. Modern adaptation of prandtl's classic lifting-line theory. *Journal of Aircraft*, 37(4):662–670, 2000.

[60] P.J. Phlips, R.A. East, and N.H. Pratt. An unsteady lifting line theory of flapping wings with application to the forward flight of birds. *J. Fluid Mech.*, 112:97–125, 1981.

[61] Ludwig Prandtl. Applications of modern hydrodynamics to aeronautics. *NACA*, 116, 1923.

[62] D.B. Quinn, G.V. Lauder, and A.J. Smits. Scaling the propulsive performance of heaving flexible panels. *J. Fluid Mech.*, 738:250–267, 2014.

[63] D.B. Quinn, G.V. Lauder, and A.J. Smits. Maximizing the efficiency of a flexible propulsor using experimental optimization. *J. Fluid Mech.*, 767:430–448, 2015.

[64] S. Ramananarivo, R. Godoy-Diana, and B. Thiria. Rather than resonance, flapping wing flyers may play on aerodynamics to improve performance. *P. Natl. Acad. Sci. USA*, 108(15):5964–5969, 2011.

[65] S. Ramananarivo, T. Mitchel, and L. Ristroph. Survival of the fastest: Evolving wings for flapping flight. *Bulletin of the American Physical Society*, 59, 2014.

[66] T. Reichert. *Kinematic Optimization in Birds, Bats and Ornithopters.* Ph.D. thesis, Inst. for Aerospace Studies, Univ. of Toronto, 2011.

[67] L. Ristroph and S. Childress. Stable hovering of a jellyfish-like flying machine. *J. R. Soc. Interface*, 11(92):20130992, 2014.

[68] V. Rokhlin. Rapid solution of integral equations of classical potential theory. *J. Comput. Phys.*, 60(2):187–207, 1985.

[69] D.T. Roper, S. Sharma, R. Sutton, and P. Culverhouse. A review of developments towards biologically inspired propulsion systems for autonomous underwater vehicles. *Proceedings of the Institution of Mechanical Engineers, Part M: Journal of Engineering for the Maritime Environment*, 225(2):77–96, 2011.

[70] K.V. Rozhdestvensky and V.A. Ryzhov. Aerohydrodynamics of flapping-wing propulsors. *Prog. Aerosp. Sci.*, 39(8):585–633, 2003.

[71] Y. Saad. A flexible inner-outer preconditioned GMRES algorithm. *SIAM J. Sci. Comput.*, 14(2):461–469, 1993.

[72] Y. Saad. *Iterative methods for sparse linear systems.* Siam, 2003.

[73] Y. Saad and M.H. Schultz. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Stat. Comp.*, 7(3):856–869, 1986.

[74] P.G. Saffman. *Vortex dynamics.* Cambridge university press, 1992.

[75] R. Servadei and E. Valdinoci. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.*, 389(2):887–898, 2012.

[76] M. Sfakiotakis, D.M. Lane, and J.B.C. Davies. Review of fish swimming modes for aquatic locomotion. *Oceanic Engineering, IEEE Journal of*, 24(2):237–252, 1999.

[77] M. Sfakiotakis, D.M. Lane, and J.B.C. Davies. Review of fish swimming modes for aquatic locomotion. *IEEE J. Oceanic Eng.*, 24(2):237–252, 1999.

[78] J.K. Shang, S.A. Combes, B.M. Finio, and R.J. Wood. Artificial insect wings of diverse morphology for flapping-wing micro air vehicles. *Bioinspir. Biomim.*, 4(3):036002, 2009.

[79] M.J. Shelley and J. Zhang. Flapping and bending bodies interacting with fluid flows. *Annu. Rev. Fluid Mech.*, 43:449–465, 2011.

[80] K. Shoele and Q. Zhu. Performance of a wing with nonuniform flexibility in hovering flight. *Phys. Fluids*, 25(4):041901, 2013.

[81] I.S. Sokolnikoff and R.D. Specht. *Mathematical theory of elasticity*, volume 83. McGraw-Hill New York, 1956.

[82] J. Song, H. Luo, and T.L. Hedrick. Wing-pitching mechanism of hovering ruby-throated hummingbirds. *Bioinspir. Biomim.*, 10(1):016007, 2015.

[83] S.E. Spagnolie, L. Moret, M.J. Shelley, and J. Zhang. Surprising behaviors in flapping locomotion with passive pitching. *Phys. Fluids*, 22(4):041903–041903, 2010.

[84] D. Tam. Flexibility increases lift for passive fluttering wings. *J. Fluid Mech.*, 765:R2, 2015.

[85] H. Tanaka, J.P. Whitney, and R.J. Wood. Effect of flexural and torsional wing flexibility on lift generation in hoverfly flight. *Integr. Comp. Biol.*, 2011.

[86] J.L. Tangorra, S.N. Davidson, I.W. Hunter, P.G.A. Madden, G.V. Lauder, H. Dong, M. Bozkurttas, and R. Mittal. The development of a biologically inspired propulsor for unmanned underwater vehicles. *Oceanic Engineering, IEEE Journal of*, 32(3):533–550, 2007.

[87] G.K. Taylor, R.L. Nudds, and A.L.R. Thomas. Flying and swimming animals cruise at a strouhal number tuned for high power efficiency. *Nature*, 425(6959):707–711, 2003.

[88] B. Thiria and R. Godoy-Diana. How wing compliance drives the efficiency of self-propelled flapping flyers. *Phys. Rev. E*, 82(1):015303, 2010.

[89] Fang-Bao Tian, Hu Dai, Haoxiang Luo, James F Doyle, and Bernard Rousseau. Fluid–structure interaction involving large deformations: 3d simulations and applications to biological systems. *J. Comput. Phys.*, 258:451–469, 2014.

[90] T.C.T. Ting. Some identities and the structure of ni in the stroh formalism of anisotropic elasticity. *Quarterly of applied mathematics*, pages 109–120, 1988.

[91] M.S. Triantafyllou, G.S. Triantafyllou, and D.K.P. Yue. Hydrodynamics of fishlike swimming. *Annu. Rev. Fluid Mech.*, 32(1):33–53, 2000.

[92] Francesco Giacomo Tricomi. *Integral equations*, volume 5. Courier Corporation, 1957.

[93] I.H. Tuncer and M. Kaya. Optimization of flapping airfoils for maximum thrust and propulsive efficiency. *AIAA J.*, 43(11):2329–2336, 2005.

[94] P.J. Van den Hout, K.J. Mathot, L.R.M. Maas, and T. Piersma. Predator escape tactics in birds: linking ecology and aerodynamics. *Behav. Ecol.*, 21(1):16–25, 2010.

[95] V. van Ginneken, E. Antonissen, U.K. Müller, R. Booms, E. Eding, J. Verreth, and G. van den Thillart. Eel migration to the sargasso: remarkably high swimming efficiency and low energy costs. *J. Exp. Biol.*, 208(7):1329–1335, 2005.

[96] E.A. van Nierop, S. Alben, and M.P. Brenner. How bumps on whale flippers delay stall: an aerodynamic model. *Phys. Rev. Lett.*, 100(5):054502, 2008.

[97] N. Vandenberghe, S. Childress, and J. Zhang. On unidirectional flight of a free flapping wing. *Phys. Fluids*, 18(1):014102–014102, 2006.

[98] N. Vandenberghe, J. Zhang, and S. Childress. Symmetry breaking leads to forward flapping flight. *J. Fluid Mech.*, 506:147–155, 2004.

[99] J.J. Videler, E.J. Stamhuis, and G.D.E. Povel. Leading-edge vortex lifts swifts. *Science*, 306(5703):1960–1962, 2004.

[100] CPALMS Perspectives Videos. Using mathematics to optimize wing design. `http://www.cpalms.org/Public/PreviewResourceUpload/Preview/%20155780`, 2015.

[101] S. Vogel. *Life in moving fluids: the physical biology of flow.* Princeton University Press, 1994.

[102] Z.J. Wang. Dissecting insect flight. *Annu. Rev. Fluid Mech.*, 37:183–210, 2005.

[103] D. Weihs. Hydromechanics of fish schooling. *Nature*, 241:290–291, 1973.

[104] J.P. Whitney and R.J. Wood. Conceptual design of flapping-wing micro air vehicles. *Bioinspiration & biomimetics*, 7(3):036001, 2012.

[105] S. J. Wright and J. Nocedal. *Numerical optimization*, volume 2. Springer New York, 1999.

[106] T. Wu. Swimming of a waving plate. *J. Fluid Mech.*, 10(03):321–344, 1961.

[107] T.Y. Wu. Fish swimming and bird/insect flight. *Annu. Rev. Fluid Mech.*, 43:25–58, 2011.

[108] B. Yin and H. Luo. Effect of wing inertia on hovering performance of flexible flapping wings. *Phys. Fluids*, 22(11):111902, 2010.

[109] Y. Yuan. A modified BFGS algorithm for unconstrained optimization. *IMA J. Numer. Anal.*, 11(3):325–332, 1991.