1 Introduction

My research focus is applied mathematics broadly. This includes the construction of mathematical models, and the development of analytical, asymptotic, and numerical techniques to examine them. Most of my work concerns modeling interactions between fluids and structures, especially in contexts motivated by geophysical or biological questions. Not only are these problems of great interest to applied mathematics, but also to diverse fields such as physics, biology, geology, marine sciences, and engineering, which often leads to fruitful collaborations. By their nature, such systems tend to be dynamic and nonlinear, necessitating the variety of approaches that I use. In these disciplines, invaluable information can be gained from experiments, and most of my research ties to experiments in some way; several of my research projects developed in close cooperation with experimentalists, while other projects help shed light on previous experimental results. I derive the greatest pleasure from combining sound analytical theory with modern computational techniques, i.e. ‘hybrid’ methods, in order to address challenging open questions or to make new predictions. In this statement, I describe a few of my recent and future research projects. The first two examples are geophysical - fluid-mechanical erosion and stratified fluid flows - while the second two are biologically relevant - complex fluids and flapping propulsion.

2 Geophysical fluid dynamics:

Erosion by flowing fluids and dynamics of stratified fluids

Air and water flows can transform landscapes by eroding solid material, typically rock, soil, or sediment. Conversely, the evolution of these landscapes can alter the behavior of such flowing fluids, and ultimately this feedback process helps create a variety of morphologies that can be observed in nature — from branched stream networks, to streamlined rock formations known as yardangs [1, 2]. Our research team set out to understand the underlying shape-flow interaction by first considering a simplified setting. We asked how would an erodible body evolve when placed in a steady, unidirectional fluid flow. Experiments of cylindrical clay bodies in flowing water revealed some unexpected features. Figures 1 (a) and (b) show that as an initially circular body erodes, it morphs into a quasi-triangular shape with front angle pointing into the flow. This form persists as the body continues to shrink. These angular features came as a surprise given the conventional view of erosion as a smoothing process, and to gain an understanding of this I developed a mathematical model based on the following simple ingredients:

1. A constitutive law with local erosion rate proportional to fluid shear stress.
2. A boundary-layer flow model to determine the shear stress along the body.

As published in PNAS, this bare-bones model allowed us to capture essential features seen in our experiments [3]. First, by seeking geometries which would maintain their shape during erosion, we were able to predict the unique angular form to which the body’s front converges. Furthermore, the model led us to predict that the body’s area would vanish with a distinctive 4/3-power law in time, as was confirmed by our experimental measurements. Ultimately, the combination of theory and experiments revealed the underlying principle that, in long time, erosion tends to reshape a body so as to evenly distribute the shear stress along its front surface.

In recent work published in Physics of Fluids, I extended these ideas to develop a fully dynamic, computational model for describing the evolution of arbitrarily shaped bodies during erosion [4].
Figure 1: The erosion of a cylindrical clay body in fast flowing water (a) at an early time when the body is circular and (b) at a later time when the body has morphed into a triangular shape. (c) The computational model decomposes the fluid flow into an inviscid outer flow (light gray), a boundary layer flow (dark gray), and a wake (blank). The bottom plot shows a numerically determined boundary-layer flow profile.

Rather than resolving in detail the complicated flows seen in Figs. 1 (a) and (b), the model exploits the simple overall structure of the flow by decomposing it into an inviscid outer component, a boundary-layer component, and a wake (illustrated in Fig. 1 (c)). By coupling these elements, the model retains the nonlocal and nonlinear character of the full system, while also being computationally tractable. Here I would like to describe this model in some detail.

First, outside of the boundary layer and the wake, the inviscid outer flow can be described by a velocity potential \( \phi \) satisfying

\[
\nabla^2 \phi = 0 \text{ in the outer region,} \tag{1}
\]

\[
\nabla \phi \cdot \hat{n} = 0 \text{ on the body and free streamlines,} \tag{2}
\]

\[
|\nabla \phi| = 1 \text{ on the free streamlines.} \tag{3}
\]

In the model, so-called free streamlines extend backwards from the body to separate the wake from the outer-flow region (see Fig. 1 (c)), and the outer flow couples to this wake structure through the boundary conditions shown above. The elliptic nature of Laplace’s equation implies nonlocal influences (both upstream and downstream), while the linearity of the PDE makes it relatively simple to compute. Determining the outer flow furnishes a surface slip velocity \( U(s) = \nabla \phi \cdot \hat{s} \), which couples to the nonlinear Prandtl boundary layer equations that govern the inner flow,

\[
\frac{u}{\partial s} + v \frac{\partial u}{\partial n} - \nu \frac{\partial^2 u}{\partial n^2} = \frac{UU'}{2}, \tag{4}
\]

\[
\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0. \tag{5}
\]

Numerically solving these equations (subject to boundary conditions) permits us to determine the shear stress on the body and then evolve the interface forward in time. Figure 2 (a) shows that during this simulated erosion, an initially circular body morphs into a similar angular shape as that seen in the experiments. Once this geometry develops, evolution proceeds self-similarly in time, as the body maintains its shape while scaling down in size. Moreover, the simulation allows us to see directly that the shear stress becomes more uniformly distributed along the body’s surface as time progresses (Fig. 2 (b)), confirming the idea of tendency towards a uniform-shear state. In our recent work, we used this model to address questions such as “How robust is the attraction towards the uniform-shear morphology?” , “How do small-scale geometric perturbations evolve?” , and “Does erosion lead to drag minimization?” [4]. Our group is currently extending similar ideas to the process of flow-driven dissolution, where we find different shape dynamics [5, 6]. In addition, I am mentoring graduate student Matt Alpert as we examine erosion at the micro-scale using accurate boundary integral methods (for my previous work on these methods see [7]).

Also belonging to the category of geophysical flows is my graduate work on density stratified fluids, which for the sake of brevity, I will describe more briefly. Here, the motivation comes from natural processes where stratified fluid flows play a crucial role, such as the mixing of heat and
Figure 2: Simulated erosion of an initially circular 2D body. (a) Computed interfaces at equally spaced time intervals and color coded in time. (b) The shear stress becomes more evenly distributed along the body’s front as time progresses.

salinity in our oceans, the formation of weather patterns, and the natural carbon cycle [8, 9]. In these settings the density layering is typically horizontal, however, under certain conditions, for example when material passes through a sharp stratification [9, 10, 11], the layering can become vertical as a result of viscous entrainment. In work published in the Journal of Fluid Mechanics, I used experiments and theory to examine such vertically-layered flow configurations, including their dynamic formation and their stability [12, 13]. In this system, linear stability analysis gives rise to a unique eigenvalue problem sharing features with both the classic Orr-Sommerfeld system (viscous and homogeneous) as well as the Taylor-Goldstein system (inviscid and stably stratified). Interestingly, this analysis led us to discover a class of long-wave instabilities that could be excited by the vertical layering, even in the limit of vanishing Reynolds number.

3 Complex fluids: the weak-coupling method

Many fluids encountered in biological and industrial settings are comprised of multiple components, for example microscopic polymers dissolved in water or oil. In these systems, macroscopic flows stretch the polymers, causing them to store elastic energy which will eventually be released back into the fluid. This feature, known as viscoelasticity, leads to a variety of interesting unsteady behaviors, while the multiple scales that are present makes such fluids challenging to model and simulate. In work recently published in the Journal of Non-Newtonian Fluid Mechanics, I developed a novel method to accurately compute viscoelastic fluid flows coupled to immersed bodies [14]. The starting point of the method is the governing Oldroyd-B equations,

\[- \nabla p + \Delta u + \beta \nabla \cdot \sigma_p = 0, \tag{6}\]
\[\frac{D}{Dt} \sigma_p - (\nabla u \cdot \sigma_p + \sigma_p \cdot \nabla u^T) + \frac{1}{Wi^{-1}}(\sigma_p - I) = 0. \tag{7}\]

Here, the fluid flow \( u \) nonlinearly couples to a polymeric stress \( \sigma_p \) that arises from deformed microscopic polymers. The dimensionless parameter \( \beta \) specifies the strength of this coupling, while the Weissenberg number \( Wi \) quantifies how long elastic energy can be stored in the polymers. For this system, the main numerical difficulties are due to stiffness; as can be seen in Eq. (7), elastic stresses can grow exponentially in time at points in the fluid where eigenvalues of \( \nabla u \) exceed \( O(Wi^{-1}) \). In most situations this exponential growth is not sustained, as Lagrangian fluid parcels are eventually transported to regions where \( \| \nabla u \| \) is small. Nonetheless, the temporary exponential growth can give rise to numerical instabilities which, once initiated, persist [15]. My weak-coupling method (WCM) exploits a parameter regime in which the polymer concentration is dilute and thus the coupling parameter, \( \beta \), is small. This limit describes a class of solutions that are widely used in experimental studies and go by the name ‘Boger fluids’ [16]. Through a regular perturbation series in \( \beta \), the WCM removes the nonlinear coupling between \( u \) and \( \sigma_p \) at leading order, and then accounts for the coupling at \( O(\beta^3) \), resulting in efficient and numerically stable computations.

In our recently published work, we applied the WCM to the archetype problem of a spherical body settling through a viscoelastic medium, and demonstrated that the method accurately captures
known unsteady behavior (see Fig. 3). Importantly, the WCM extends the range of computable \( Wi \) by more than an order of magnitude when compared to traditionally methods. In addition, the WCM provides valuable analytical insight into the scaling of high elastic stresses that develop in the wake of the body, consistent with experimental observations of a so-called birefringent strand. More recently, I supervised undergraduate Dustin Hill, who applied the WCM to the problem of a slender rod settling through a viscoelastic medium [17]. Here, viscoelasticity strongly influences the settling orientation of the rod, causing it to eventually align vertically. Interestingly, this is exactly the opposite behavior as occurs in a Newtonian fluid.

4 Current and future work

The thesis of Dustin Hill relied heavily on the ideas of the weak-coupling method (WCM), and in the future I envision applying these ideas to a range of problems concerning interactions between structures and complex fluids. Some biological examples include collective behavior of microorganisms moving through viscoelastic media [18] [19], or transport of mucus from beating cilia. Meanwhile, some industrially motivated problems include how to model the behavior of particle-laden or bubbly flows in the presence of viscoelasticity [20].

In a separate project, I am examining the role of flexibility in flapping propulsion, which is important for flying and swimming animals. A number of recent studies indicate that flexibility can substantially improve locomotive performance under certain conditions. Due to the complexity of such systems — elastic structures interacting with high-Reynolds-number fluid flows — these conclusions have come mainly from experiments and direct numerical simulations [21], with few useful analytical results. I have found that it is possible to employ small-amplitude asymptotics to determine the fluid forces produced by a flapping wing, and to couple these forces to elastic deformations of the wing. Though limited to small amplitude, the results robustly recover a range of behaviors including: improved performance of a flexible wing at small frequencies, peak performance at a so-called resonant frequency, and underperformance at yet higher frequencies [22, 23, 24]. From my perspective, having analytical results for a simplified problem such as this is fundamental to our understanding of wing flexibility in the complicated settings of flying/swimming animals, where wing/fins may have irregular geometry and non-uniform distributions of mass and flexibility, and where the background flow itself may be irregular, unsteady, and three-dimensional.

References


