

Outline of a theory of turbulent shear flow

By W. V. R. MALKUS

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

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SUMMARY

In this paper the spatial variations and spectral structure of steady-state turbulent shear flow in channels are investigated without the introduction of empirical parameters. This is made possible by the assumption that the non-linear momentum transport has only stabilizing effects on the mean field of flow. Two constraints on the possible momentum transport are drawn from this assumption: first, that the mean flow will be statistically stable if an Orr-Sommerfeld type equation is satisfied by fluctuations of the mean; second, that the smallest scale of motion that can be present in the spectrum of the momentum transport is the scale of the marginally stable fluctuations of the mean. Within these two constraints, and for a given mass transport, an upper limit is sought for the rate of dissipation of potential energy into heat. Solutions of the stability equation depend upon the shape of the mean velocity profile. In turn, the mean velocity profile depends upon the spatial spectrum of the momentum transport. A variational technique is used to determine that momentum transport spectrum which is both marginally stable and produces a maximum dissipation rate. The resulting spectrum determines the velocity profile and its dependence on the boundary conditions. Past experimental work has disclosed laminar, 'transitional', logarithmic and parabolic regions of the velocity profile. Several experimental laws and their accompanying constants relate the extent of these regions to the boundary conditions. The theoretical profile contains each feature and law that is observed. First approximations to the constants are found, and give, in particular, a value for the logarithmic slope (von Kármán's constant) which is within the experimental error. However, the theoretical boundary constant is smaller than the observed value. Turbulent channel flow seems to achieve the extreme state found here, but a more decisive quantitative comparison of theory and experiment requires improvement in the solutions of the classical laminar stability problem.

INTRODUCTION

The significant difficulty encountered in the theoretical study of fluid turbulence is the non-linearity of the equations of motion. The classical inquiries (e.g. Goldstein 1938), which are concerned with gross spatial

features of the flow, avoid dealing with non-linearity by deducing macroscopic laws from the observations. The more recent statistical inquiries (e.g. Batchelor 1953), which are mainly restricted to homogeneous fields of motion, deal with non-linearity by various hypotheses concerning the transfer of energy from one scale of motion to another. The emphasis in these latter studies is on the local random aspects of turbulence and on those regions of the spectrum of turbulence far removed from external energy sources. Hence the spatial variations and spectral structure of momentum and heat transport, which are intimately coupled to the energy sources, receive little theoretical attention.

In this paper the emphasis is on the statistical organization of turbulent fields rather than on their statistical randomness. Hence the momentum and heat transport of a steady-state turbulent field and the dependence of these transports on boundary conditions are of primary concern. However, this will not be a detailed mechanistic study. Indeed, there is little hope that exact time-dependent solutions of the non-linear equations describing turbulence can be found. Here, as in past work, an assumption specifying the role of the non-linear processes will be made. This particular assumption is a negative one, specifying what the non-linear processes can not do. In this way the class of possible turbulent fields is restricted. The determination of an extreme member of this class reduces the mathematical problem to tractable linear form without the introduction of empirical parameters. This absence of empirical parameters permits a rigorous appraisal of the validity of the assumption. Indeed, if the quantitative solutions to the problem are to compare favourably with the observations, the assumption must embody all the significant restraints imposed by the equations of motion.

The particular physical situation to be discussed is the fully turbulent flow in channels. This flow has been studied experimentally for many years. Hence it is an excellent testing ground for theoretical models. The model evolved here has its origin in conventional laminar stability theory. Therefore a brief restatement of the conclusions of this theory is of value.

The stability of a time-independent state of a fluid field can be established by study of the growth or decay of superimposed infinitesimal disturbances. If any disturbance releases a larger amount of potential energy from the pressure field than it dissipates through the action of viscosity, it will grow. The mathematical formulation of this kind of problem usually leads to a linear characteristic value equation. The external conditions which delimit the range of stability of the initial time-independent state of the fluid are determined by the minimum characteristic value. In simple shearing flow a disturbance can abstract energy from the pressure field only through the agency of viscous stress. Hence, in this case, viscosity plays the double role of permitting the growth of disturbances and also dissipating them.

When a disturbance achieves finite amplitude, its non-linear interaction with the initial motion alters both this motion and the disturbance.

First-order stability theory tells one nothing of this alteration, while inclusion of the non-linear interaction in the problem leads to such intractable mathematics that even second-order effects have seldom been investigated. However, the physics of the non-linear effects is fairly clear. The disturbance must grow until its alteration of the initial motion reduces the rate of release of potential energy to the rate of dissipation of this energy. If the disturbance is an aperiodic turbulent motion rather than another steady laminar motion, the equalization of rates of energy release and dissipation must still occur on the average. In either the laminar or turbulent case, alteration of the initial motion is accomplished by the transport of heat or momentum, whichever was responsible for the instability. Hence the most important aspects of the non-linear interaction of disturbance and initial motion are those advective terms which readjust the energy sources of the disturbance.

In the new 'stability' that is achieved, the finite-amplitude disturbance can be looked upon as the stabilizing sink of energy for the destabilizing energy source in the pressure field. It will be assumed that this is so on the average even in the fully developed steady-state turbulence, i.e. that the mean non-linear momentum transport terms in the equations of motion are entirely stabilizing, and that the only energy source for those disturbances which enter into the mean momentum transport is the pressure field of the mean motion.

In the first section, a limiting 'statistical stability' condition on the spatial structure of the mean momentum transport is found as a consequence of this assumption. The momentum transport is then resolved into a spatial spectrum whose smallest scale of motion is the scale of the smallest statistically unstable motion. The determination of the particular spectrum satisfying these stability constraints and the boundary conditions, and also releasing the largest amount of potential energy, leads to an unusual formal problem. In this problem the non-constant coefficients of a linear characteristic value equation are varied under constraints to determine a minimum least characteristic value, also the corresponding minimum least characteristic function and the optimum form of the non-constant coefficients.

The second section outlines a variational approach designed to find these optimal functions and constants; i.e. to find the momentum transport spectrum and mean velocity profile which lead to a marginally stable mean field of maximum dissipation rate. Considerable use is made of the knowledge acquired in the past thirty years about solutions of the laminar stability equation for shearing flow. A first approximation to the spectrum is found, and a comparison is made between this theoretical extreme and the observations.

The concluding section describes other more detailed tests of this work by comparison with recent channel flow data, e.g. the observation of a smallest scale of motion. An attempt to rationalize the basic assumption of the theory raises questions concerning the uniqueness of solutions to the equations of motion, while the apparent existence of extreme states in

turbulent fields suggests an explicit statistical mechanical origin. Finally, the application of this approach to quite different turbulent phenomena is outlined.

1. FORMULATION OF A TRACTABLE PROBLEM

This is a study of the steady-state fully evolved three-dimensional turbulence observed between fixed parallel surfaces. Here 'fully evolved' will mean both that the Reynolds number (based on the channel half-width and the average velocity) is far above its value for marginal stability, and that there are no variations of moments of the fluctuating field in the downstream direction. In figure 1 the coordinate frame is chosen so that the x direction is the direction of the mean velocity U , which in turn is some unknown function of the cross-stream direction z . The 'mean' will be defined here as an average in the y direction (perpendicular to the plane of figure 1). It is assumed that the effect of the distant y boundaries on real flows does not influence the nature of the motion far from those boundaries. Hence the remainder of this work deals with parallel surfaces of infinite extent.

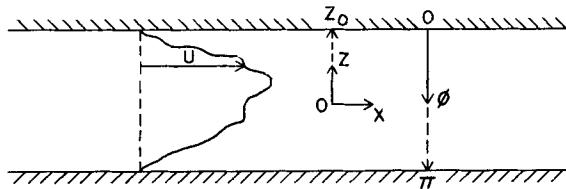


Figure 1. Coordinates chosen to describe flow between parallel surfaces.

It is also assumed in general that compressibility, heating effects and non-linear viscous effects are unimportant in the experimental situation. Therefore the equations of motion and continuity are

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v} \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0, \quad (1.1)$$

where D/Dt is the substantial derivative, P the pressure, ρ the fixed density, ν the constant kinematic viscosity, and \mathbf{v} the velocity vector.

A first concern will be the determination of statistical stability constraints imposed on the steady-state flow by these equations.

As implied by the name, statistical stability means the constancy in time of some space (or ensemble) average of the field of motion. Here an investigation will be made of the time dependence of the quantity

$$\bar{\mathbf{v}}(x, z, t) = \left[\frac{1}{2y_0} \int_{-y_0}^{+y_0} \mathbf{v} dy \right]_{y_0 \rightarrow \infty} = \mathbf{v} - \mathbf{v}'. \quad (1.2)$$

Future use of the bar will indicate the mean with respect to y , as in (1.2). Expanding (1.1) with the aid of (1.2), one may write

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} + \frac{\partial \mathbf{v}'}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \bar{\mathbf{v}} + \mathbf{v}' \cdot \nabla \mathbf{v}' = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \bar{\mathbf{v}} + \nu \nabla^2 \mathbf{v}',$$

and

$$\nabla \cdot \bar{\mathbf{v}} + \nabla \cdot \mathbf{v}' = 0, \quad (1.3)$$

The mean value of (1.3) is then

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} + \overline{\mathbf{v}' \cdot \nabla \mathbf{v}'} = -\frac{1}{\rho} \nabla_2 \bar{P} + \nu \nabla_2^2 \bar{\mathbf{v}}$$

and

$$\nabla_2 \cdot \bar{\mathbf{v}} = 0, \quad (1.4)$$

where ∇_2 indicates the two-dimensional (x, z) del operator. Subtracting (1.4) from (1.3), one obtains

$$\frac{\partial \mathbf{v}'}{\partial t} + (\bar{\mathbf{v}} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \bar{\mathbf{v}}) + (\mathbf{v}' \cdot \nabla \mathbf{v}' - \overline{\mathbf{v}' \cdot \nabla \mathbf{v}'}) = -\frac{1}{\rho} (\nabla P - \nabla_2 \bar{P}) + \nu \nabla^2 \mathbf{v}',$$

and

$$\nabla \cdot \mathbf{v}' = 0. \quad (1.5)$$

Now if the mean flow is to be stable and independent of x , i.e. if $\bar{\mathbf{v}} = U(z)\mathbf{i}$, then it follows from (1.4) that

$$\left[-\nu \frac{\partial^2 U}{\partial z^2} \right] \mathbf{i} + \left[\frac{1}{\rho} \frac{\partial P}{\partial x} \right] \mathbf{i} + \left[\frac{1}{\rho} \frac{\partial P}{\partial z} \right] \mathbf{k} = -\left[\frac{\partial}{\partial z} \overline{w'u'} \right] \mathbf{i} - \left[\frac{\partial}{\partial z} \overline{w'^2} \right] \mathbf{k}, \quad (1.6)$$

where \mathbf{i} and \mathbf{k} are unit vectors in the x and z directions respectively, and u' and w' are the components of \mathbf{v}' in the x and z directions respectively. Equation (1.6) is, of course, the usual equation for the mean flow. Integrating the \mathbf{k} equation, one finds that $\bar{P}/\rho + \overline{w'^2} = \bar{P}_0/\rho$, where \bar{P}_0 is the pressure at either fixed surface; and since $\partial \overline{w'^2}/\partial x = 0$ (no variation of mean flow moments in the x direction), then $\partial \bar{P}/\partial x = \partial \bar{P}_0/\partial x$. Integrating the \mathbf{i} equation, one finds that, since there must be no torque on the fluid as a whole,

$$-\left(\frac{1}{\rho} \frac{\partial \bar{P}_0}{\partial x} \right) z = \nu \beta + \overline{w'u'} = \tau = \tau_0 \frac{z}{z_0}, \quad (1.7)$$

where $\beta = -\partial U/\partial z$, and $\tau_0 = -\left(\frac{1}{\rho} \frac{\partial \bar{P}_0}{\partial x} \right) z_0$ is the stress per unit mass at the fixed surface. Equation (1.7) specifies the spatial relation of the mean gradient and the mean momentum transport. Thus it is basic to the following work.

However, the stability of any particular mean flow is not assured unless an arbitrary mean disturbance disappears in a finite time. If we take

$$\bar{\mathbf{v}} = U(z)\mathbf{i} + \epsilon \bar{\mathbf{v}}_1(x, z, t) + \epsilon^2 \bar{\mathbf{v}}_2(x, z, t) + \dots,$$

and

$$\bar{P} = \bar{P}(x, z) + \epsilon \bar{P}_1(x, z, t) + \epsilon^2 \bar{P}_2(x, z, t) + \dots, \quad (1.8)$$

where ϵ is a constant parameter, then $\bar{\mathbf{v}}_1, \bar{P}_1, \bar{\mathbf{v}}_2, \bar{P}_2$, etc. must all decay in time for an arbitrary choice of ϵ . From (1.4), the zeroth-order terms in ϵ give (1.6). The first-order ϵ terms in (1.4) are

$$\frac{\partial \bar{\mathbf{v}}_1}{\partial t} + U \frac{\partial \bar{\mathbf{v}}_1}{\partial x} + \left(\bar{\mathbf{v}}_1 \cdot \mathbf{k} \frac{\partial U}{\partial z} \right) \mathbf{i} - \nu \nabla_2^2 \bar{\mathbf{v}}_1 + \frac{1}{\rho} \nabla_2 \bar{P}_1 = -[\Delta \overline{\mathbf{v}' \cdot \nabla \mathbf{v}'}]$$

and

$$\nabla_2 \cdot \bar{\mathbf{v}}_1 = 0, \quad (1.9)$$

where $[\Delta \overline{\mathbf{v}' \cdot \nabla \mathbf{v}'}]$ signifies any first-order change that occurs in the mean non-linear stress due to the complex interaction in (1.5) of $\bar{\mathbf{v}}_1$ and \bar{P}_1 with

the field of finite fluctuations. If arbitrary $\bar{\mathbf{v}}_1$ and \bar{P}_1 are to decay in time, equation (1.9) places restrictions on the form and amplitude of $U(z)$, and hence, through (1.7), on the form and amplitude of $\overline{w'u'}$.

At this point in the analysis, the mechanistic assumption that $\overline{\mathbf{v}' \cdot \nabla \mathbf{v}'}$ plays only a stabilizing role is introduced. The conditions placed on stable forms of $U(z)$ and $\overline{w'u'}$ will be at least as severe as the actual case if the assumed stabilizing effect of $[\Delta \mathbf{v}' \cdot \nabla \mathbf{v}']$ is neglected. Then, with $[\Delta \mathbf{v}' \cdot \nabla \mathbf{v}']$ set equal to zero, equation (1.9) assumes the familiar linear form of the equation for the disturbance of a time-independent shearing flow. Here, however, the problem is to find the entire class of U , for given boundary conditions, which lead to marginally stable $\bar{\mathbf{v}}$ and \bar{P} .

Taking the curl of (1.9) to remove the pressure term, and assuming that

$$\bar{\mathbf{v}}_1 \cdot \mathbf{k} = \Phi(z) \exp i(\alpha/z_0)[x - (cU_m)t], \quad (1.10)$$

one may write

$$\begin{aligned} \left(\frac{U}{U_m} - c \right) \left(\frac{\partial^2 \Phi}{\partial (z/z_0)^2} - \alpha^2 \Phi \right) - \frac{\partial^2 (U/U_m)}{\partial (z/z_0)^2} \Phi \\ = \frac{i}{\alpha R} \left(\frac{\partial^4 \Phi}{\partial (z/z_0)^4} - 2\alpha \frac{\partial^2 \Phi}{\partial (z/z_0)^2} + \alpha^4 \Phi \right), \end{aligned} \quad (1.11)$$

where α/z_0 is the downstream wave number of the disturbance of the mean, cU_m is its phase velocity, U_m is the z -averaged mean velocity, and $R = z_0 U_m / \nu$ is the Reynolds number. Equation (1.11) is called the Orr–Sommerfeld equation (Orr 1906–7, Sommerfeld 1908), and has been studied for many years to determine initial instability criterion for a given laminar U . Its appearance in a problem of fully turbulent flow is due in part to the present special assumption and in part to the vanishing of explicit interaction terms between $\bar{\mathbf{v}}$ and \mathbf{v}' in (1.4).

In his study of equation (1.11), Heisenberg (1924) has shown that for a given R some smallest scale of motion (largest α) is unstable. (This is also established in equation (2.41) of this paper.) In addition if $\overline{w'u'}$ is entirely stabilizing, it cannot produce motions requiring greater destabilizing force than that available in the mean flow. Hence a second consequence of the assumption is that the spectrum of $\overline{w'u'}$ cannot contain a scale of motion smaller than twice the scale of the smallest marginally stable motion according to (1.11). (We specify ‘twice’ since $\overline{w'u'}$ is a product of the fluctuation velocities.)

This restriction on the spectrum of $\overline{u'w'}$ is, through (1.7), a restriction on the possible spatial structure of U . Hence the class of possible U is doubly restricted by the marginal solutions of the stability equation (1.11). Within this class, for a given U_m , one function leads to the release of more potential energy per unit time than any other. This function can be thought of as the ‘most stable’ in the sense that any alteration of it will increase the potential energy available to disturbances. Ignorance of the detailed mechanisms of momentum transport permits one to hope that this extreme state is approached in the observed turbulence. Hence in the remainder of this section a tractable formal problem for the extreme is posed.

In the steady state, the rate of release of potential energy equals the rate at which mechanical work is done by the shearing stresses, and this equals the rate at which energy is dissipated into heat by the action of viscosity. Hence

$$(\beta\tau)_m = \nu \left[\overline{\left(\frac{\partial \mathbf{v}}{\partial x}\right)^2} + \overline{\left(\frac{\partial \mathbf{v}}{\partial y}\right)^2} + \overline{\left(\frac{\partial \mathbf{v}}{\partial z}\right)^2} \right]_m, \quad (1.12)$$

where $(\partial \mathbf{v}/\partial x)^2 = (\partial \mathbf{v}/\partial x) \cdot (\partial \mathbf{v}/\partial x)$, etc., the subscript m again indicates an average value in the z direction, and

$$(\beta\tau)_m = \tau_0(\beta z/z_0)_m = \tau_0 \beta_0 \quad (1.13)$$

from (1.7). Also

$$\beta_0 = \frac{1}{2z_0} \int_{-z_0}^{+z_0} \beta \frac{z}{z_0} dz = - \frac{zU}{2z_0} \Big|_{-z_0}^{+z_0} + \frac{1}{2z_0} \int_{-z_0}^{+z_0} \frac{U}{z_0} dz = \frac{U_m}{z_0}. \quad (1.14)$$

Hence the total rate of dissipation per unit mass, $\tau_0 U_m$, will be a maximum for a given U_m when τ_0 is a maximum. In a non-dimensional form, a maximum for $(z_0/\nu)(\tau_0/U_m)$ is sought for a given $(z_0/\nu)U_m = R$. However, it is convenient to invert this problem and seek the minimum Reynolds number associated with a marginally stable solution to (1.11), holding $(z_0/\nu)(\tau_0/U_m)$ constant. This latter quantity is proportional to the ratio of the actual dissipation rate and the dissipation rate which would result from a laminar parabolic flow with the same U_m . Indeed, it is readily shown that

$$(\tau_0)_{\text{laminar}} = 3\nu\beta_0. \quad (1.15)$$

Hence the ratio

$$\frac{\tau_0}{(\tau_0)_{\text{laminar}}} = \frac{1}{3} \left(\frac{z_0}{\nu} \right) \left(\frac{\tau_0}{U_m} \right) = \frac{\beta(z_0)}{3\beta_0} \quad (1.16)$$

will be held constant during the minimization of R . The other variables in (1.11) for which optimum values will be sought are α , c and the mean velocity U . There are, however, certain geometric and boundary constraints on U which first must be determined.

One notes that since $u = v = w = 0$ and $\partial w/\partial z = 0$ (from $\nabla \cdot \mathbf{v} = 0$) at the boundary surfaces z_0 and $-z_0$, then

$$u'w' = \frac{\partial u'w'}{\partial z} = \frac{\partial^2 u'w'}{\partial z^2} = 0,$$

at z_0 and $-z_0$. Hence, from (1.7), we have

$$\begin{aligned} \beta &= \frac{1}{\nu} \left(\tau_0 \frac{z}{z_0} - \overline{u'w'} \right) = - \frac{\partial U}{\partial z}; \quad \text{thus} \quad \beta(z_0) = \frac{\tau_0}{\nu}, \quad \beta(-z_0) = - \frac{\tau_0}{\nu}; \\ \frac{\partial \beta}{\partial z} &= \frac{\tau_0}{\nu z_0} - \frac{1}{\nu} \frac{\partial \overline{u'w'}}{\partial z} = - \frac{\partial^2 U}{\partial z^2}; \quad \text{thus} \quad \frac{\partial \beta}{\partial z}(z_0) = \frac{\partial \beta}{\partial z}(-z_0) = \frac{\tau_0}{\nu z_0}; \\ \frac{\partial^2 \beta}{\partial z^2} &= - \frac{1}{\nu} \frac{\partial^2 \overline{u'w'}}{\partial z^2} = - \frac{\partial^3 U}{\partial z^3}; \quad \text{thus} \quad \frac{\partial^2 \beta}{\partial z^2}(z_0) = \frac{\partial^2 \beta}{\partial z^2}(-z_0) = 0; \end{aligned} \quad (1.17)$$

and also $U = 0$ at z_0 and $-z_0$.

Yet another constraint on U arises from certain general conclusions regarding the stability of solutions of (1.11). It has been found (Heisenberg

1924, Lin 1945–46) that a necessary, but not sufficient, requirement for stable solutions at large Reynolds numbers is that the curvature of the mean flow be always of the same sign. A type of instability called ‘inflexional’ can result if this is not the case. Hence one further requires of U that

$$-\frac{\partial^2 U}{\partial z^2} = \text{const.} \times d^2, \quad (1.18)$$

where d is a real quantity. It is convenient to define the dimensionless quantity

$$\mathcal{U} = \frac{U}{U_m}, \quad (1.19)$$

and choose the constant in (1.18) so that

$$\frac{\partial^2 \mathcal{U}}{d\phi^2} = -\frac{12}{\pi^2} \frac{\tau_0}{(\tau_0)_L} d^2 \equiv -K d^2, \quad (1.20)$$

where $\phi = \frac{1}{2}\pi \cdot (1 - z/z_0)$ (see figure 1). In terms of ϕ , the boundary conditions on d^2 , for equations (1.17), may be expressed as follows:

$$\left. \begin{aligned} (a) \quad & \left(\frac{\partial d^2}{\partial \phi} \right)_0 = \left(\frac{\partial d^2}{\partial \phi} \right)_\pi = 0; \\ (b) \quad & (d^2)_0 = (d^2)_\pi = 1; \\ (c) \quad & \frac{1}{\pi} \int_0^\pi d^2 d\phi = 1; \\ (d) \quad & \frac{2}{\pi^2} \int_0^\pi \phi d^2 d\phi = 1, \quad \text{or} \quad \int_{-z_0}^{+z_0} z d^2 dz = 0. \end{aligned} \right\} \quad (1.21)$$

The definitional constraint $\pi^{-1} \int_0^\pi \mathcal{U} d\phi = 1$ may be written

$$\frac{2}{\pi^3} \int_0^\pi \phi^2 d^2 d\phi = 1 - \frac{4}{\pi^2 K}. \quad (1.22)$$

Now (d) of (1.21) is just the requirement that d^2 be symmetric; accordingly, equation (1.22) may be written

$$\frac{1}{K} = \int_0^{\frac{1}{2}\pi} \left(\phi - \frac{\phi^2}{\pi} \right) d^2 d\phi. \quad (1.23)$$

From (1.7) we have

$$\frac{\overline{u'w'}}{\tau_0} = \frac{2}{\pi} \int_0^\phi (d^2 - 1) d\phi. \quad (1.24)$$

Hence the spatial spectrum of the momentum transport is determined by d^2 . We have also

$$\mathcal{U} = K \left\{ \frac{1}{2}\pi\phi - \phi \int_0^\phi d^2 d\phi + \int_0^\phi \phi d^2 d\phi \right\}. \quad (1.25)$$

The final constraint on U is due to the requirement that the field of transporting motions, $\overline{u'w'}$, has some smallest scale, eventually to be determined by the stability equation. This constraint is introduced by

the requirement that d has some smallest scale n_0 , that is

$$d = \sum_{n=0}^{n_0} Y_n \psi_n(\phi), \tag{1.26}$$

where $\psi_n(\phi)$ is some real orthogonal set of functions satisfying the appropriate boundary and symmetry conditions, and the Y_n are arbitrary real amplitudes.

The formal linear problem may now be stated: First, for any U satisfying $-(\partial^2 U / \partial z^2) \geq 0$, find from the Orr-Sommerfeld equation (1.11) the smallest scale of motion and the relation between R , α and c for marginal stability. Second, expressing U in terms of the d from (1.18) and taking some convenient set ψ_n , vary Y_n , subject to the constraints (1.21) and (1.23), to find the minimum R for constant K . This analysis is undertaken in the next section.

2. A VARIATIONAL SOLUTION TO THE STABILITY PROBLEM

The stability equation (1.11) has been studied since 1908. A recent interesting inquiry (Lin 1945-46) contains, as a partial list, seventy references. A first purpose in all these papers was to establish the Reynolds number for the initial instability of laminar flow. Only an outline of this theory will be given here.

For a given velocity profile U , the Orr-Sommerfeld equation determines a relation $c = c(\alpha, R)$ for each solution Φ . For R and α real, and c complex, this single relation can be separated into a real and imaginary part. For marginal stability the imaginary part of c is zero and this determines $R = R(\alpha)$, the curve of marginal stability. For each point on this curve the real part of the relation $c = c(\alpha, R)$ determines the corresponding value of c . No complete solution has yet been found. However, asymptotic solutions of equation (1.11) for large values of αR have proved adequate in the study of initial laminar instability (and should be much better for the turbulence problem since αR is orders of magnitude larger). Recent quantitative work rests on the relations*:

$$\left(\Phi_3 / \frac{\partial \Phi_3}{\partial \phi} \right)_{\phi=0} = \phi_c F(\eta), \tag{2.1}$$

where Φ_3 is the highly oscillatory part of the asymptotic Φ , and ϕ_c is that value of ϕ at which $U = c$;

$$\eta = [(1 + \lambda)(1 - \gamma)^{1/3}] \cdot (\frac{1}{4} \pi^2 K)^{-2/3} c(\alpha R)^{1/3}, \tag{2.2}$$

where

$$\gamma = \frac{2}{\pi} \int_0^{\phi_c} d^2 d\phi, \quad \frac{\lambda}{1 + \lambda} = \frac{1}{\phi_c} \int_0^{\phi_c} \left(\frac{2}{\pi} \int_0^{\phi} d^2 ds \right) d\phi; \tag{2.3}$$

and

$$F(\eta) = \frac{\int_{+\infty}^{-\eta} d\xi' \int_{+\infty}^{\xi'} \xi^{1/2} d\xi H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right]}{-\eta \int_{+\infty}^{-\eta} \xi^{1/2} d\xi H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right]} \equiv 1 - \frac{1}{J(\eta)}.$$

* Lin's equations (6.28), (6.29), and (12.32) correspond to (2.1), (2.2), and (2.4) in the notation used here.

The function $F(\eta)$ was first numerically calculated by Tietjens. Lin has improved and extended this calculation, of which relevant results appear in table 1 below.

η	2.6	2.8	3.0	3.2	3.4	3.6	3.8
$F_i(\eta)$	0.07203	0.12220	0.17391	0.22520	0.27193	0.30705	0.32130
$J_i(\eta)$	0.31558	0.46043	0.54872	0.58082	0.56401	0.51074	0.43560
$F_r(\eta)$	0.52773	0.49952	0.46456	0.41947	0.36110	0.28802	0.20352

Table 1

A first relation imposed by the boundary conditions on the several parts of the asymptotic Φ is written

$$J_i(\eta) = -\frac{1}{2}\pi^2 K(1-2\lambda) \left[\frac{\mathcal{U} \partial^2 \mathcal{U} / \partial \phi^2}{(\partial \mathcal{U} / \partial \phi)^3} \right]_{\phi=\phi_c} = 2\phi_c d_c^2 [(1+\lambda)(1-\gamma)]^{-3} = v(c), \quad (2.4)$$

where the primary contribution to $v(c)$ is due to the pole of equation (1.11) at $U = c$, and where λ , as will be justified *a posteriori*, is assumed to be very small. The value of ϕ_c is determined from (2.4) at the first interception of $v(c)$ and $J_i(\eta)$. Equations (2.1), (2.2) and (2.4) can then be used to determine αR as a function of c ; hence they determine $(\alpha R)_{\min}$ and the corresponding c . An important characteristic of the $R(\alpha)$ curve, used since Heisenberg first developed this approach, is that the value of R at $(\alpha R)_{\min}$ is very close to the minimum value of R . This property of $R(\alpha)$ will also be used here to find R_{\min} . The real task of recent work has been the evaluation of α from the complicated second relation imposed by the boundary conditions on the asymptotic Φ . This will be avoided here through the use of an integral condition for stability discussed later in this section.

Turning now to the turbulence problem, one notes that the scale of motion in Φ is determined by the imaginary part of the derivative of Φ . The largest value of this derivative will occur in the sharp shear zone at the boundary. Hence

$$n_0 = \left| \left(\frac{1}{\Phi_3} \frac{\partial \Phi_3}{\partial \phi} \right)_i \right|_{\phi=0}. \quad (2.5)$$

Then, from (2.1), one may write

$$\xi_c = \frac{1}{F_i(\eta)}. \quad (2.6)$$

where $\xi_c = n_0 \phi$. Equation (2.4) is now written

$$(d^2/n_0)_c = \frac{1}{2} [(1+\lambda)(1-\gamma)]^3 J_i(\eta) F_i(\eta). \quad (2.7)$$

To determine $(\alpha R)_{\min}$, Heisenberg first observed that for $\partial(\alpha R)/\partial c = 0$, one has from (2.2) and (2.4)

$$\eta J'_i(\eta) = cv'(c), \quad (2.8)$$

and

$$(\alpha R)_{\min} = \left(\frac{1}{4}\pi^2 K \right)^2 \left(\frac{\eta_0}{c} \right)^3 1/(1+\lambda)^3 (1-\gamma), \quad (2.9)$$

where η_0 is that η which satisfies (2.8), and where the small spatial variation of $(1 + \lambda)^3(1 - \gamma)$ has been neglected. Hence, if the Y_n in the expansion of d^2 , equation (1.26), are constrained to satisfy (2.7) and (2.8), the resulting profile will be both marginally stable and continuously at $(\alpha R)_{\min}$, no matter what further variations of Y_n are made.

Now a convenient orthonormal set $\psi_n(\phi)$ which automatically satisfies (a) and (d) of (1.21) is

$$\psi_n(\phi) = \sqrt{2} \cos 2n\phi, \quad \psi_0(\phi) = 1, \quad (2.10)$$

where n is any integer. Hence, to satisfy (b) and (c) of (1.21), we must have

$$\sum_{n=0}^{\frac{1}{2}n_0} Y_n^2 = 1, \quad (2.11)$$

and

$$\left(\sum_{n=0}^{\frac{1}{2}n_0} Y_n \right)^2 = \frac{1}{2}. \quad (2.12)$$

Equation (1.23) may be written

$$\frac{1}{K} = \sum_{n=0}^{\frac{1}{2}n_0} \sum_{m=0}^{\frac{1}{2}n_0} B_{nm} Y_n Y_m, \quad (2.13)$$

where

$$B_{nm} = \int_0^{\pi/2} \left(\phi - \frac{\phi^2}{\pi} \right) [\cos 2(n+m)\phi + \cos 2(n-m)\phi] d\phi.$$

Also from (1.24) and (2.10) the momentum transport spectrum is

$$\frac{\overline{u'w'}}{\tau_0} = \sum_{n=1}^{n_0} T_n \sin 2n\phi,$$

where

$$T_n = \frac{1}{2n} \sum_{x=-\frac{1}{2}n_0}^{\frac{1}{2}n_0} Y_{|x|} Y_{|n-x|}. \quad (2.14)$$

It is now possible to write down a complete set of equations optimizing Y_n for a minimum $(\alpha R)_{\min}$. However, not only is this a most complicated problem, but its solution will be as approximate as the stability criteria. Thus, to establish the more significant physical consequences of the extreme state, the spectrum Y_n will be approximated to roughly the same extent that Φ is in equation (2.1). This approximation of the optimum Y_n will be done in two ways: First, only the consequences of Y_n at large n_0 will be considered, so that in expansions in powers of $1/n_0$ only the first term will be kept. Second, rather than optimize each Y_n separately, optimization will be made between sets of Y_n which satisfy the conditions (2.7), (2.8), (2.10) and (2.11).

Although other trial functions may converge more rapidly to the optimum dissipation rate, a simple power series has many mathematical advantages. Hence Y_n is expanded as

$$Y_n = \frac{A}{\sqrt{n_0}} \left[1 + \frac{C_1}{(n_0/2)} n + \frac{C_2}{(n_0/2)^2} n^2 + \frac{C_3}{(n_0/2)^3} n^3 + \dots \right]. \quad (2.15)$$

To have even one free parameter for a variational optimization, Y_n must

have five free parameters. Therefore Y_n in (2.15) will be terminated at the n^3 term, leaving A , n_0 , C_1 , C_2 and C_3 to be determined. From (2.15) and (1.26) one finds that

$$d = \frac{A}{(2n_0)^{1/2}} \left[\left\{ 1 + \frac{\sin(n_0+1)\phi}{\sin\phi} \right\} + \frac{C_1}{(n_0/2)} \left\{ \left(\frac{n_0}{2} + 1 \right) \frac{\sin(n_0+1)\phi}{\sin\phi} - \frac{1(1-\cos(n_0+1)\phi)}{\sin^2\phi} \right\} + \frac{C_2}{(n_0/2)^2} \left\{ \left(\frac{n_0}{2} + 1 \right)^2 \frac{\sin(n_0+1)\phi}{\sin\phi} + \frac{(n_0+3)\cos(n_0+2)\phi}{2} - \frac{1\sin(n_0+3)\phi}{2\sin^3\phi} \right\} + \frac{C_3}{(n_0/2)^3} \left\{ \left(\frac{n_0}{2} + 1 \right)^3 \frac{\sin(n_0+1)\phi}{\sin\phi} + \frac{3}{2} \left(\frac{n_0+1}{2} \right)^2 \frac{\cos(n_0+1)\phi \cos\phi}{\sin^2\phi} - \frac{3}{4} \left(\frac{n_0+1}{2} \right) \frac{\sin(n_0+1)\phi(1+\cos^2\phi)}{\sin^3\phi} + \frac{1(1+2\cos^2\phi)}{4\sin^4\phi} - \frac{1\cos(n_0+1)\phi \cos\phi(5+\cos^2\phi)}{8\sin^4\phi} \right\} \right], \quad (2.16)$$

where the various sums, $\sum_{n=0}^{n_0} n \cos 2n\phi$, etc., have been performed by the calculus of finite differences. In keeping with the requirement of large n_0 , it is convenient to divide d^2 into a function describing its complicated transition region near the boundary and a function which is important away from the boundary. These are

$$(d^2 n_0)_{\phi \gg 1/n_0} = \frac{A^2}{2} \left[1 + C_0 \frac{\sin(n_0+1)\phi}{\sin\phi} \right]^2, \quad (2.17)$$

and

$$(d^2/n_0)_{\phi \sim 1/n_0} = \frac{A^2}{2\xi^2} \left[C_0 \sin\xi + C_1 \left\{ -\frac{(1-\cos\xi)}{\xi} \right\} + C_2 \left\{ 2\frac{\cos\xi}{\xi} - \frac{2\sin\xi}{\xi^2} \right\} + C_3 \left\{ 3\frac{\cos\xi}{\xi} - 6\frac{\sin\xi}{\xi^2} + 6\frac{(1-\cos\xi)}{\xi^3} \right\} \right] = \frac{A^2}{2\xi^2} L^2(\xi), \quad (2.18)$$

where $\hat{C}_0 = 1 + C_1 + C_2 + C_3$, and $\xi = n_0 \phi$. Hence equation (2.7) becomes

$$AL(\xi_c) = [(1+\lambda)(1-\gamma)]^{3/2} [J_i(\eta_0)/F_i(\eta_0)]^{1/2}. \quad (2.19)$$

Moreover, if one neglects the variations of $[(1+\lambda)(1-\gamma)]^{3/2}$ with ξ_c , equation (2.8) becomes

$$\frac{L'(\xi_c)}{L(\xi_c)} = \frac{1}{2} F_i(\eta_0) \left[1 + \frac{\eta_0 J_i'(\eta_0)}{J_i(\eta_0)} \right]. \quad (2.20)$$

Equation (2.12) can be summed to give

$$2 = -C_1 - \frac{2}{3}C_2 - \frac{1}{2}C_3 \quad (2.21)$$

to the zeroth order in $1/n_0$, while (2.11) is summed as

$$2 = A^2 \left\{ 1 + C_1 + \frac{1}{3}(2C_2 + C_1^2) + \frac{1}{2}(C_1 C_2 + C_3) + \frac{1}{5}(2C_1 C_3 + C_2^2) + \frac{1}{3}C_2 C_3 + \frac{1}{7}C_3^2 \right\}. \quad (2.22)$$

Then equations (2.6), (2.19), (2.20), (2.21) and (2.22) are five relations between six independent variables A , C_1 , C_2 , C_3 , η_0 and ξ_c . A choice for ξ_c , say, determines the other five. Actually, λ and γ are small quadratic functions of these independent variables, from equation (2.3). However, to a zeroth approximation they may be assumed to be zero, their values then computed for each ξ_c and these values used for a first approximation of A , C_1 , C_2 , C_3 at each ξ_c .

Before resolving these algebraic relations the remainder of the problem will be discussed. From (2.9) we have

$$(\alpha R)_{\min} = \frac{1}{2}\pi K^2 \left(\frac{n_0 \eta_0}{K \xi_c} \right)^3 \frac{1}{(1-\gamma)}; \quad (2.23)$$

i.e. $C(1+\lambda) = \frac{1}{2}\pi K/\xi_c$. Hence the dependence of n_0/K on

$$\xi_c = \xi_c(A, C_1, C_2, C_3, \eta_0)$$

must be found in order to determine that ξ_c (say ξ_c) which makes $(\alpha R)_{\min}$ a minimum. Following this, to determine the relation between n_0 and R , α must be determined. Now α , the down-stream wave number, bears some (probably fixed) relation to $\frac{1}{2}\pi n_0$, the cross-stream wave number, say

$$\frac{1}{2}\pi n_0 = r\alpha. \quad (2.24)$$

(If $r = 1$, the smallest 'eddy' would be circular; if $r = 10$ the smallest 'eddy' would be a 10 to 1 ellipse with major axis parallel to the boundary.) Then from (2.6) and (2.23) it follows that

$$R_{\min} = \frac{r}{(1-\gamma)} \left(\frac{\eta_0}{\xi_c} \right)^3 \left(\frac{n_0}{K} \right) n_0. \quad (2.25)$$

Hence a determination of the ratio r establishes the relation $R = R(n_0)$.

To interpret the various formulae in conventional experimental terms, it is convenient to define a 'boundary Reynolds number':

$$R_B = \frac{sU_s}{\nu}, \quad (2.26)$$

where s is a distance proportional to z_0/n_0 , say $s = z_0/n_0$, and U_s is the velocity that would be due to the initial gradient at that distance from the boundary. So defined, equation (2.26) may be written as

$$R_B = \frac{\pi^2 RK}{4 n_0^2}, \quad \text{or} \quad n_0 = \frac{z_0}{\nu} \frac{1}{\sqrt{R_B}} U_s, \quad \text{or} \quad \frac{n_0}{K} = \frac{\pi^2}{4} \frac{1}{\sqrt{R_B}} \frac{U_m}{U_s}, \quad (2.27)$$

where $U_s = \sqrt{\tau_0}$ is the conventional 'friction velocity', in terms of which most data have been described. From (2.27) and (2.25) we have

$$R_B = \frac{\pi^2}{4} \frac{r}{(1-\gamma)} \left(\frac{\eta_0}{\xi_c} \right)^3. \quad (2.28)$$

Finally, to find n_0/K and the velocity profile as functions of

$$\xi_c = \xi_c(A, C_1, C_2, C_3),$$

equations (1.23) and (1.26) may be integrated using (2.17) and (2.18). From

(1.26), (2.17), and (2.18), it follows that

$$\begin{aligned} \frac{n_0}{K} = \int_0^{\frac{1}{2}\pi} \left(\phi - \frac{\phi^2}{\pi} \right) d^2 n_0 d\phi &\doteq \int_0^{\frac{1}{2}\pi - \epsilon} \left(\frac{\pi}{4} - \frac{\theta^2}{\pi} \right) (d^2 n_0)_{\phi > > 1/n_0} d\theta + \\ &+ \int_0^{n_0 \epsilon} \left(\xi - \frac{\xi^2}{n_0 \pi} \right) \left(\frac{d^2}{n_0} \right)_{\phi \sim 1/n_0} d\xi, \end{aligned} \quad (2.29)$$

where $\theta = \frac{1}{2}\pi - \phi$, and $1/n_0 \ll \epsilon \ll 1$. For large n_0 ,

$$n_0/K = \frac{1}{2} A^2 \left[\frac{1}{2} C_0^2 (\ln \Gamma n_0 + 1) + \pi^2/12 + C \right], \quad (2.30)$$

where $\Gamma = 1.78+$ is Euler's constant, and

$$C = \frac{(2 \ln 2 + 1)}{2} (1 - C_0^2) + (2 \ln 2 - 1) \frac{C_2}{15} (2C_0 - 3 + \frac{1}{6} C_2) \quad (2.31)$$

after some condensation with the help of (2.21). From (1.25) and (2.18), the velocity field near the boundary may be determined in terms of sine and cosine integrals:

$$\frac{n_0}{K} \eta l = \frac{1}{2} \pi \xi - \xi \int_0^{\xi} \left(\frac{d^2}{n_0} \right)_{\phi \sim 1/n_0} ds + \int_0^{\xi} s \left(\frac{d^2}{n_0} \right)_{\phi \sim 1/n_0} ds.$$

After rising linearly, as $\frac{1}{2} \pi \xi$, from the boundary, $n_0 \eta l/K$ turns sharply through a 'transition' region to its asymptotic value

$$n_0 \eta l/K = \frac{1}{2} A^2 \left[\frac{1}{2} C_0^2 (\ln 2 \Gamma \xi + 1) + \pi^2/12 + C \right]. \quad (2.32)$$

From (1.23) and (2.17), direct integration gives

$$\frac{n_0}{K} (\eta l_{\max} - \eta l) = \frac{1}{2} A^2 \left[\frac{1}{2} \theta^2 - \frac{1}{2} C_0^2 \ln \cos \theta \right]. \quad (2.33)$$

Now combining (2.30), (2.32) and (2.33) with (2.27), one may write a theoretical form for the gross features of channel flow beyond the 'transition' region:

$$\begin{aligned} \frac{U}{U_\tau} = \sqrt{R_B} \left(\frac{AC_0}{\pi} \right)^2 \left[\ln \frac{(z_0 - z) U_\tau}{\nu} + \left\{ \left(1 + \ln \frac{\pi \Gamma}{\sqrt{R_B}} \right) + \right. \right. \\ \left. \left. + \frac{2}{C_0^2} \left(\frac{\pi^2}{12} + C \right) \right\} \right], \end{aligned} \quad (2.34)$$

$$\frac{U_{\max} - U}{U_\tau} = \sqrt{R_B} \left(\frac{AC_0}{\pi} \right)^2 \left[\ln \frac{1}{\cos \theta} + \frac{\theta^2}{C_0^2} \right]; \quad (2.35)$$

$$\begin{aligned} \frac{U_m}{U_\tau} = \sqrt{R_B} \left(\frac{AC_0}{\pi} \right)^2 \left[\ln \frac{z_0 U_\tau}{\nu} + \left\{ \left(1 + \ln \frac{\Gamma}{\sqrt{R_B}} \right) + \right. \right. \\ \left. \left. + \frac{2}{C_0^2} \left(\frac{\pi^2}{12} + C \right) \right\} \right]. \end{aligned} \quad (2.36)$$

Those readers familiar with the experimental literature will have recognized a 'velocity defect law' in equation (2.35) and a 'logarithmic boundary law' in (2.34). The usual empirical constants in these experimental laws depend here on the determination of $\xi_c = \xi_c(A, C_1, C_2, C_3, \eta_0)$ and the ratio r .

Returning to the algebraic problem of finding $\xi_c = \xi_c(A, C_1, C_2, C_3, \eta_0)$ from (2.6), (2.19), (2.20), (2.21) and (2.22), one notes that, for a particular

ξ_c , η_0 is determined from (2.6) and table 1. Resolving the other four equations, it is found that

$$C_1 = -2 - \left(\frac{2}{3} + \frac{e_3}{2}\right)C_2 + \frac{1}{2}Q; \quad C_2 = QG_1 - G_2; \quad C_3 = Q + e_3 C_2, \tag{2.37}$$

where

$$Q = \frac{e_1}{A} + e_2 = -\frac{q_2}{q_1} \pm \left[\left(\frac{q_2}{q_1}\right)^2 - \frac{q_3}{q_1} \right]^{1/2},$$

$$q_1 = -e_5 G_1 - e_4 G_1^2 - \frac{11}{420} + \frac{2}{e_1^2},$$

$$q_2 = \frac{1}{2}\{e_5 G_2 + e_6 G_1\} + e_4 G_1 G_2 + \frac{1}{15} - \frac{2e_2}{e_1^2},$$

$$q_3 = -e_6 G_2 - e_4 G_2^2 - \frac{1}{3} + \frac{2e_2^2}{e_1^2};$$

also

$$G_1 = \frac{e'_1 - 1}{e_3 - e'_3}, \quad G_2 = \frac{e_1 e_2 - e'_2}{e_3 - e'_3},$$

$$e_1 = \frac{[(1 + \lambda)(1 - \gamma)]^{3/2} [J_i(\eta_0)/2F_i(\eta_0)]^{1/2}}{(d_3 - \frac{1}{2}d_1)},$$

$$e_2 = \frac{(2d_1 - d_0)}{(d_3 - \frac{1}{2}d_1)}, \quad e_3 = \frac{(\frac{2}{3}d_1 - d_2)}{(d_3 - \frac{1}{2}d_1)},$$

$$e_4 = \left(\frac{2}{135} + \frac{7}{180}e_3 + \frac{11}{420}e_3^2\right), \quad e_5 = \left(\frac{7}{180} + \frac{11}{210}e_3\right),$$

$$e_6 = \left(\frac{1}{9} + \frac{2}{15}e_3\right), \quad e'_1 = \frac{\frac{1}{2}F_i(\eta)[1 + \eta_0 J'(\eta_0)](d_3 - \frac{1}{2}d_1)}{J(\eta_0)(d'_3 - \frac{1}{2}d'_1)}$$

$$e'_2 = \frac{(2d'_1 - d'_0)}{(d'_3 - \frac{1}{2}d'_1)}, \quad e'_3 = \frac{(\frac{2}{3}d'_1 - d'_2)}{(d'_3 - \frac{1}{2}d'_1)};$$

and

$$L(\xi_c) \equiv (d_0 + C_1 d_1 + C_2 d_2 + C_3 d_3), \quad L'(\xi_c) \equiv (d'_0 + C_1 d'_1 + C_2 d'_2 + C_3 d'_3).$$

For each chosen ξ_c , the quantity $(n_0/K)(\eta_0/\xi_c)$ of equation (2.23) may be found from these relations and (2.30). However, the restriction that the first interception of $J_i(\eta)$ and $v(c)$, equation (2.4), determines ξ_c sets an upper limit on the ξ_c that may be chosen. Table 2 lists the computed values for $(n_0/K)(\eta_0/\xi_0)$, the corresponding values of γ and λ , and the maximum value of

$$I = AL(\xi)\xi^{1/2}\{[1 + \lambda(\xi)] \cdot [1 - \gamma(\xi)]\}^{3/2}/[J_i(\eta_0)]^{1/2}, \quad \xi < \xi_c,$$

at several ξ_c on either side of this upper limit. When $I = 1$, $J_i(\eta)$ marginally

ξ_c	$\left(\frac{n_0}{K} \cdot \frac{\eta_0}{\xi_c}\right)_{n_0=10^3}$	$\left(\frac{n_0}{K} \cdot \frac{\eta_0}{\xi_c}\right)_{n_0=10^4}$	γ	λ	$I(\xi)$	ξ
6.00	14.87	30.14	0.095	0.045	0.846	0.93 π
350° = 6.1086	14.40	28.78	0.121	0.065	1.042	0.97 π
2 π	13.84	27.80	0.183	0.114	1.375	1.05 π

Table 2

intercepts $v(c)$ at the smaller ξ . Hence, from a graph of table 2, the minimum value of $(n_0/K)(\eta_0/\xi_c)$ occurs at $\xi_c = \underline{\xi}_c = 6.081$. Substitution of this ξ_c into the relations (2.37) gives

$$C_2 = 25.67, \quad Q = -1.630, \quad A^2 = 11.43;$$

and
$$\frac{\eta_0}{\underline{\xi}_c} = 0.4875, \quad C_0^2 = 2.308, \quad C + \frac{\pi^2}{12} = -1.908, \quad (2.38)$$

for
$$\gamma = 0.114, \quad \lambda = 0.059.$$

The ratio r can be determined from the second relation imposed by the boundary conditions on the asymptotic Φ . (Equation (2.4) was the first relation.) However, this is a difficult task involving sums of series of integrals of functions of the optimum velocity profile. Instead, an integral condition for stability due to Synge (1938) will be used. Synge observed that multiplying the Orr-Sommerfeld equation (1.11) by $\Phi^* d\phi$, integrating from 0 to π and adding the resulting equation to its own complex conjugate leads to
$$I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 = -\alpha R \mathcal{U}'(0) I_3 - \alpha R c_i (I_1^2 + \alpha^2 I_0^2), \quad (2.39)$$

where $I_0^2 = \int_0^\pi \Phi \Phi^* d\phi, \quad I_1^2 = \int_0^\pi \Phi' \Phi^* d\phi, \quad I_2^2 = \int_0^\pi \Phi'' \Phi^* d\phi,$

$$I_3 = \frac{1}{2} i \int_0^\pi (\mathcal{U}'/\mathcal{U}'(0)) (\Phi' \Phi^* - \Phi \Phi^{*\prime}) d\phi,$$

and the prime denotes differentiation with respect to ϕ . When Φ is stable, we have $c_i \leq 0$; hence

$$1/I_0^2 \int_0^\pi \left[1 + \frac{\mathcal{U}'(0)}{\alpha^3} R \left\{ \frac{i}{2} \frac{\mathcal{U}'}{\mathcal{U}'(0)} \left(\frac{\Phi'}{\Phi} - \frac{\Phi^{*\prime}}{\Phi^*} \right) + \frac{2}{\alpha^2} \left(\frac{\Phi' \Phi^{*\prime}}{\Phi \Phi^*} \right) \right\} \right] \Phi \Phi^* d\phi + \left(\frac{I_2}{\alpha^2 I_0} \right)^2 \geq 0. \quad (2.40)$$

A sufficient condition for stability is that the integral in (2.40) be greater than zero. If then $\Phi'/\Phi = A + iB$, this integral is positive definite when

$$[\mathcal{U}'(0)R]^2 = 8\alpha^4 M, \quad M = \frac{\int_0^\pi (A^2 + B^2) \Phi \Phi^* d\phi \int_0^\pi \Phi \Phi^* d\phi}{\left[\int_0^\pi (\mathcal{U}'/\mathcal{U}'(0)) B \Phi \Phi^* d\phi \right]^2}. \quad (2.41)$$

Synge noted that, since $\int B^2 \Phi \Phi^* d\phi \int \Phi \Phi^* d\phi \geq \left[\int B \Phi \Phi^* d\phi \right]^2$, M in (2.41) must be greater than one. Hence, $[\mathcal{U}'(0)R]^2 = 8c_i^4$ sets a lower limit on α for stable Φ . However, a closer condition can be set by using the asymptotic Φ to estimate M . Now at the boundary, $\phi = 0, A(0) \geq F_r(\eta_0)$ and $B(0) = F_i(\eta_0)$. From (2.38) and table 1, $F_r(\eta_0) = 0.465$ and

$$F_i(\eta_0) = 0.174.$$

Also from table 1 one sees that $F_r(\eta_0)/F_i(\eta_0)$ increases for smaller η (i.e. larger ϕ) through the range where $\mathcal{U}'/\mathcal{U}'(0)$ is large. Hence, in a conservative estimate, we have

$$\int_0^\pi (A^2 + B^2) \Phi \Phi^* d\phi \geq \left\{ 1 + \left[\frac{A(0)}{B(0)} \right]^2 \right\} \int_0^\pi B^2 \Phi \Phi^* d\phi,$$

and

$$M \geq \left[1 + \left(\frac{0.465}{0.174} \right)^2 \right] \doteq 8.$$

Since the value of M will enter final formulae only as $M^{1/2}$, this represents

only a slight improvement on Synge's condition. An extension of the preceding analysis shows that $I_2/\alpha^2 I_0$ and $I_1/\alpha I_0$ approach zero for large α and larger R ; hence using (2.41), we have

$$C_i \rightarrow - \left[\frac{\alpha}{R} \left(1 - \frac{2}{\alpha^2} \right)^2 \right] \rightarrow 0.$$

Equation (2.41) then becomes as accurate an estimate of the marginally stable α as one could hope to attain by the tedious method mentioned earlier.

Finally, equating the R in (2.41) and (2.23), one obtains

$$r = (1 - \gamma)^{1/3} \left(\frac{\xi_c}{\eta_0} \right) (8M)^{1/6} = 3.94 \quad (2.42)$$

for $M = 8$, $\eta_0/\xi_c = 0.4875$ and $\gamma = 0.114$ from (2.38). Inserting the numerical results of (2.38) and (2.42) into (2.28) and the constants of (2.34) and (2.35), one obtains

$$R_B = \frac{\pi^2}{4} \frac{r}{(1 - \gamma)} \left(\frac{\eta_0}{\xi_c} \right)^3 = 1.271, \quad (2.43)$$

$$\sqrt{R_B} \left(\frac{AC_0}{\pi} \right)^2 = 3.014; \quad \text{e.v.} = 3.0 \pm 0.1, \quad (2.44)$$

$$\left(1 + \ln \frac{\pi\Gamma}{\sqrt{R_B}} \right) + \frac{2}{C_0^2} \left(\frac{\pi^2}{12} + C \right) = +1.001; \quad \text{e.v.} = 1.8 \pm 0.1, \quad (2.45)$$

where e.v. means the experimental values recently determined by Laufer (1950).

The constant of equation (2.44) (inversely proportional to von Kármán's constant) is the most important in these results since it enters into every other relation. The 'logarithmic intercept' constant, equation (2.45), involves the relatively sensitive quantity C (equation (2.31)), perhaps accounting for the error in this determination. An observation of r will be discussed in the conclusion. The dissipation rate is inversely proportional to the square of U_m/U_r , equation (2.36); hence it is larger than observed due to the error in C . However, at large R the percentage error due to C is small, and the dissipation rate approaches the observed curve.

It should be emphasized that an extension of the technique outlined here involves considerable computational labour. In addition, further variational freedom for the Y_n spectrum may necessitate the use of R_{\min} rather than the approximate $(\alpha R)_{\min}$.

To summarize this section: All known gross qualitative features of turbulent flow in smooth channels result from an optimization of the dissipation rate subject to the mean stability constraints; a decisive quantitative comparison with the observations is not achieved primarily due to the mathematical difficulties encountered in finding a general solution to the Orr-Sommerfeld equation.

3. CONCLUSION

From a mathematical viewpoint the foregoing analysis is most laborious. The asymptotic solution to the Orr-Sommerfeld equation for a quite general U is described. Only then is U restricted to an optimal form.

However, the stability problem might be considerably simplified if the major restrictions on U imposed by the optimization could be made first. Some slight progress has been made along this line by performing constrained variations of U in the integral stability equations (2.39). This approach offers hope of a more complete understanding of the problem than the maze of formulae employed here permits.

On the physical side this work strongly supports the basic mechanistic assumption. Its validity can be tested by measuring the smallest scales of motion in the transport and energy spectra. Laufer (1950) has made time-spectra studies in channel flow from the boundaries to channel centre. The smallest scale of motion he detected in the energy spectrum was in the middle of the 'laminar' boundary region, and was twice as large as the observed thickness of this region. Since the energy spectrum has twice the maximum wave number of the velocity spectrum, this observation establishes that the minimum ellipticity of the smallest scale of motion is approximately four to one. Equation (2.42) gives a theoretical optimum value of 3.94. A more successful investigation of the detailed stability restrictions on the spectral transfer of energy may one day remove the assumption entirely.

Before leaving the topic of spectra, it should be noted that the spatial spectrum for the transport given by (2.14) differs in kind from the temporal spectra obtained by the experimentalist. In homogeneous turbulence, spatial and temporal spectra would be identical; but in this inhomogeneous and non-isotropic shear flow, the observed spectra are functions of position. The work presented here does not determine the temporal spectra and their spatial dependence, but does provide a framework for such an inquiry.

The apparent achievement of a statistical extreme state by the turbulent fluid raises questions of a different sort than those raised by the mechanistic assumption. It suggests that one observes the most probable of a large set of possible motions: that turbulent solutions to the equations of motion may be highly degenerate. If this is so, what determines the observed spatial and spectral structure of statistical observables, such as $\overline{v'^2}$, not studied here? One might think that degeneracy implied the equal *a priori* probability of each solution, but the infinite series of higher order small corrections to the usual equations of motion remove degeneracies and couple the macroscopic field of motion to the underlying microscopic particle motion. Hence it is proposed that the probability of occurrence of a degenerate solution is directly proportional to its degree of disorganization, or entropy. The search for a useful measure of the disorganization of irreversible fluid processes involves sufficiently different formal techniques to necessitate separate discussion. However, even the usual thermodynamic statistical mechanics suggests that a steady-state system with a fixed number of particles would have maximum 'entropy' when it has a maximum 'temperature' and when all the dynamic variables have, as nearly as possible, normal Gaussian distributions about their mean values. In steady-state shear flow, the mean temperature is a maximum relative to the boundary

temperature for a maximum rate of dissipation of potential energy into heat. Yet to achieve this first extreme the mean momentum transport must be highly organized, as studied in this paper. Hence a completely random distribution of either macroscopic or microscopic dynamic variables is not possible. A study of possible secondary extremes for partially constrained macroscopic observables of turbulent shear flow will be presented in another paper.

Finally, it is of interest that the techniques used here are of value in the study of other forms of turbulence. Many of the laminar stability equations developed for geophysical and astrophysical problems can serve as the foundation for determining the gross statistical features of their corresponding turbulent fields. Certain of these stability problems are much easier than the Orr–Sommerfeld problem. In particular this is true of turbulence due to thermal convection. With heat flux replacing momentum transport, mean temperature profile replacing the mean velocity profile, and Rayleigh's equation replacing the Orr–Sommerfeld equation, turbulent convection may be treated just as shear flow was here. Two earlier papers (Malkus 1954) report the initial results of such a study.

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