Integrable Nonlocal Nonlinear Equations

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A nonlocal nonlinear Schrödinger (NLS) equation was recently found by the authors and shown to be an integrable infinite dimensional Hamiltonian equation. Unlike the classical (local) case, here the nonlinearly induced "potential" is PT symmetric thus the nonlocal NLS equation is also PT symmetric. In this paper, new *reverse space-time* and *reverse time* nonlocal nonlinear integrable equations are introduced. They arise from remarkably simple symmetry reductions of general AKNS scattering problems where the nonlocality appears in both space and time or time alone. They are integrable infinite dimensional Hamiltonian dynamical systems. These include the reverse space-time, and in some cases reverse time, nonlocal NLS, modified Korteweg-deVries (mKdV), sine-Gordon, $(1+1)$ and $(2+1)$ dimensional three-wave interaction, derivative NLS, "loop soliton," Davey–Stewartson (DS), partially PT symmetric DS and partially reverse space-time DS equations. Linear Lax pairs, an infinite number of conservation laws, inverse scattering transforms are discussed and one soliton solutions are found. Integrable reverse space-time and reverse time nonlocal discrete nonlinear Schrödinger type equations are also introduced along with few

*This paper is dedicated to the memory of Professor David J. Benny. David Benney was on the faculty at MIT from 1959 until his retirement in 2013. One of the authors, MJA, met Benney in 1966 when he came for an informal interview to become a graduate student. Subsequently, Benney became his thesis advisor. It was a special period for the field of nonlinear waves and David Benney was a central figure. His accomplishments were enormous; they are discussed in an article published in *Studies in Applied Math* **108**, pp. 1–6 (2002). That journal volume was dedicated to him. We believe that this paper is appropriate for this dedicated volume of *Studies in Applied Mathematics* because it builds on a "Studies" paper by M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur in 1974. David Benny was always very supportive to have Soliton/Integrable Systems research papers published in "Studies." We were/are deeply grateful. Dave is and will be profoundly missed as a person and as a mathematician.

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DOI: 10.1111/sapm.12153
STUDIES IN APPLIED MATHEMATICS 139:7–59

conserved quantities. Finally, nonlocal Painlevé type equations are derived from the reverse space-time and reverse time nonlocal NLS equations.

1. Introduction

Since their fundamental discovery in 1965 by Zabusky and Kruskal, solitons have emerged as one of the most basic concepts in nonlinear sciences. Physically speaking, solitons (or previously termed solitary waves) represent robust nonlinear coherent structures that often form as a result of a delicate balance between effects of dispersion and/or diffraction and wave steepening. They have been theoretically predicted and observed in laboratory experiments in many branches of the physical, biological and chemical sciences. Examples include water waves, temporal and spatial optics, Bose–Einstein condensation, magnetics, plasma physics to name a few—see [1–5] and references therein for reviews discussing soliton applications.

From a mathematical point of view, solitons naturally arise as a special class of solutions to so-called integrable evolution equations. Such integrable systems exhibit unique mathematical structure by admitting an infinite number of constants of motion corresponding to an infinite number of conservation laws. Furthermore, by applying the inverse scattering transform (IST; cf. [6–8]), for decaying data, one can linearize the system and obtain significant information about the structure of their solutions. In many situations, one can even express these solutions in closed form.

Historically speaking, the first integrable nonlinear evolution equation solved by the method of IST was the Korteweg-deVries (KdV) equation [9]. Remarkably, it was shown that solitons corresponded to eigenvalues of the time independent linear Schrödinger equation. Soon thereafter, the concept of Lax pair [10] was introduced and the KdV equation, and others, were expressed as a compatibility condition of two linear equations. A few years later, Zakharov and Shabat [11] used the idea of Lax pair to integrate the nonlinear Schrödinger equation

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q^*(x,t), \sigma = \mp 1, \tag{1}$$

for decaying data, where $*$ is the complex conjugate, and obtain soliton solutions. Subsequently, a method to generate a class of integrable nonlinear integrable evolution equations solvable by IST was developed [12]. Soon thereafter, interest in the theory of integrability has grown significantly and many integrable nonlinear partial differential equations (PDEs) have been identified in both one and two space dimensions as well as in discrete settings. Some notable equations include the modified KdV, sine-Gordon, sinh-Gordon, coupled NLS, Boussinesq, Kadomtsev–Petviashvili, Davey-Stewartson (DS), Benjamin-Ono (BO), Intermediate Long Wave (ILW), integrable discrete NLS equations, the Toda and discrete KdV lattices, to name a few cf. [13]. In 2013, a new nonlocal reduction of the AKNS scattering problem was found [14], which gave rise to an integrable nonlocal NLS equation

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q^*(-x,t), \quad \sigma = \mp 1 \ . \tag{2}$$

Remarkably, Eq. (2) has a self-induced nonlinear "potential," thus, it is a PT symmetric equation [15]. In other words, one can view (2) as a linear Schrödinger equation

$$iq_t(x,t) = q_{xx}(x,t) + V[q,x,t]q(x,t), \tag{3}$$

with a self-induced potential $V[q,x,t] \equiv -2\sigma q(x,t)q^*(-x,t)$ satisfying the PT symmetry condition $V[q,x,t] = V^*[q,-x,t]$. We point out that PT symmetric systems, which allow for lossless-like propagation due to their balance of gain and loss [16, 17], have attracted considerable attention in recent years—see [18] and references therein for an extensive review on linear and nonlinear waves in PT symmetric systems. Equation (2) was derived in [14] with physical intuition. Recently, the nonlocal nonlinear Schrödinger (NLS) equation (2) was derived in a physical application of magnetics [19]. In [20] an integrable discrete PT symmetric "discretization" of Eq. (2) was obtained from a new nonlocal PT symmetric reduction of the Ablowitz–Ladik scattering problem [21]. In [22] the detailed IST associated with the nonlocal NLS system (2) was carried out and integrable nonlocal versions of the modified KdV and sine-Gordon equations were introduced. An extension to a $(2+1)$-dimensional integrable nonlocal NLS type equations was subsequently analyzed in [24].

These findings have triggered renewed interest in integrable systems. New types of soliton solutions have been also recently reported [25, 26]. Moreover, recently, it was proposed that the integrable nonlocal (in space) NLS equation is gauge equivalent to an unconventional system of coupled Landau–Lifshitz equations [19]. Possible application of the NLS and mKdV equations have been discussed in [27, 28] in the context of "Alice-Bob systems." In this paper, we identify new nonlocal symmetry reductions for the general AKNS scattering problem of the *reverse space-time* and *reverse time type*. Unlike the integrable PT symmetric equation (2) [14], here the symmetry reductions are nonlocal both in space and time or time alone and lead to new integrable reverse space-time nonlocal evolution equations of the nonlinear Schrödinger, modified KdV, sine-Gordon, (1+1) and (2+1) dimensional multiwave interaction (including the three-wave), derivative NLS, "loop soliton," DS, partially PT symmetric DS, and partially reverse space-time DS equations. Furthermore, discrete-time nonlocal NLS type equations are also derived. Finally, nonlocal Painlevé type equations are derived from the reverse space-time and reverse time nonlocal NLS equations.

Next, we list some of these equations (here $\sigma = \mp 1; \gamma^2 = \pm 1, \alpha, \beta, c_j, \mathbf{C}_j$ are constant).
Reverse space-time nonlocal NLS:

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(-x,-t), \tag{4}$$

Reverse space-time vector nonlocal NLS:

$$i\mathbf{q}_t(x,t) = \mathbf{q}_{xx}(x,t) - 2\sigma[\mathbf{q}(x,t)\cdot\mathbf{q}(-x,-t)]\mathbf{q}(x,t), \tag{5}$$

Reverse time nonlocal NLS:

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(x,-t), \tag{6}$$

Reverse space-time coupled nonlocal NLS – derivative NLS ($\alpha, \beta \in \mathbb{R}$):

$$q_t(x,t) = iq_{xx}(x,t) + \alpha\sigma(q^2(x,t)q(-x,-t))_x + i\beta\sigma q^2(x,t)q(-x,-t), \tag{7}$$

Real reverse space-time nonlocal nonlinear "loop soliton":

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{q_x(x,t)}{[1-\sigma q(x,t)q(-x,-t)]^{3/2}}\right) = 0, \tag{8}$$

Complex reverse space-time nonlocal mKdV:

$$q_t(x,t) + q_{xxx}(x,t) - 6\sigma q(x,t)q^*(-x,-t)q_x(x,t) = 0, q \in \mathbb{C}, \tag{9}$$

Real reverse space-time nonlocal mKdV:

$$q_t(x,t) + q_{xxx}(x,t) - 6\sigma q(x,t)q(-x,-t)q_x(x,t) = 0, q \in \mathbb{R}, \tag{10}$$

Real reverse space-time nonlocal sine-Gordon:

$$\begin{aligned} &q_{xt}(x,t) + 2s(x,t)q(x,t) = 0,\ q \in \mathbb{R}, \\ &s_x(x,t) = (q(x,t)q(-x,-t))_t, \end{aligned} \tag{11}$$

Reverse space-time nonlocal DS:

$$\begin{aligned} &iq_t(\mathbf{x},t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t) + q_{yy}(\mathbf{x},t)\right] + \sigma q^2(\mathbf{x},t)q(-\mathbf{x},-t) = \phi(\mathbf{x},t)q(\mathbf{x},t), \\ &\phi_{xx}(\mathbf{x},t) - \gamma^2\phi_{yy}(\mathbf{x},t) = 2\sigma\left[q(\mathbf{x},t)q(-\mathbf{x},-t)\right]_{xx}, \end{aligned} \tag{12}$$

Reverse time nonlocal DS:

$$\begin{aligned} &iq_t(\mathbf{x},t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t) + q_{yy}(\mathbf{x},t)\right] + \sigma q^2(\mathbf{x},t)q(\mathbf{x},-t) = \phi(\mathbf{x},t)q(\mathbf{x},t), \\ &\phi_{xx}(\mathbf{x},t) - \gamma^2\phi_{yy}(\mathbf{x},t) = 2\sigma\left[q(\mathbf{x},t)q(\mathbf{x},-t)\right]_{xx} \end{aligned} \tag{13}$$

Partially PT symmetric nonlocal DS:

$$iq_t(\mathbf{x},t)+\frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t)+q_{yy}(\mathbf{x},t)\right]+\sigma q^2(\mathbf{x},t)q^*(-x,y,t)=\phi(\mathbf{x},t)q(\mathbf{x},t),$$
$$\phi_{xx}(\mathbf{x},t)-\gamma^2\phi_{yy}(\mathbf{x},t)=2\sigma\left[q(\mathbf{x},t)q^*(-x,y,t)\right]_{xx}, \tag{14}$$

Partial reverse space-time nonlocal DS:

$$iq_t(\mathbf{x},t)+\frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t)+q_{yy}(\mathbf{x},t)\right]+\sigma q^2(\mathbf{x},t)q(-x,y,-t)=\phi(\mathbf{x},t)q(\mathbf{x},t),$$
$$\phi_{xx}(\mathbf{x},t)-\gamma^2\phi_{yy}(\mathbf{x},t)=2\sigma\left[q(\mathbf{x},t)q(-x,y,-t)\right]_{xx}, \tag{15}$$

Reverse space-time nonlocal three wave interaction with $c_3>c_2>c_1, \sigma_1\sigma_3/\sigma_2=1.$:

$$Q_{1,t}(x,t)+c_1Q_{1,x}(x,t)=\sigma_3 Q_2(-x,-t)Q_3(-x,-t),\quad \sigma_3=\pm 1$$
$$Q_{2,t}(x,t)+c_2Q_{2,x}(x,t)=-\sigma_2 Q_1(-x,-t)Q_3(-x,-t),\quad \sigma_2=\pm 1$$
$$Q_{3,t}(x,t)+c_3Q_{3,x}(x,t)=\sigma_1 Q_1(-x,-t)Q_2(-x,-t),\quad \sigma_1=\pm 1 \tag{16}$$

Multidimensional reverse space-time nonlocal three wave interaction with distinct $\mathbf{C}_j$, $j=1,2,3, \sigma_1\sigma_3/\sigma_2=1$:

$$Q_{1,t}(\mathbf{x},t)+\mathbf{C}_1\cdot\nabla Q_1(\mathbf{x},t)=\sigma_3 Q_2^*(-\mathbf{x},-t)Q_3^*(-\mathbf{x},-t),\quad \sigma_3=\pm 1$$
$$Q_{2,t}(\mathbf{x},t)+\mathbf{C}_2\cdot\nabla Q_2(\mathbf{x},t)=-\sigma_2 Q_1^*(-\mathbf{x},-t)Q_3^*(-\mathbf{x},-t),\quad \sigma_2=\pm 1$$
$$Q_{3,t}(\mathbf{x},t)+\mathbf{C}_3\cdot\nabla Q_3(\mathbf{x},t)=\sigma_1 Q_1^*(-\mathbf{x},-t)Q_2^*(-\mathbf{x},-t),\quad \sigma_1=\pm 1 \tag{17}$$

Reverse time nonlocal discrete NLS:

$$i\frac{dQ_n(t)}{dt}=Q_{n+1}(t)-2Q_n(t)+Q_{n-1}(t)$$
$$-\sigma Q_n(t)Q_n(-t)\left[Q_{n+1}(t)+Q_{n-1}(t)\right], \tag{18}$$

Reverse discrete-time nonlocal discrete NLS:

$$i\frac{dQ_n(t)}{dt}=Q_{n+1}(t)-2Q_n(t)+Q_{n-1}(t)$$
$$-\sigma Q_n(t)Q_{-n}(-t)\left[Q_{n+1}(t)+Q_{n-1}(t)\right], \tag{19}$$

In the above, $\mathbf{x}$ represents (x,y) and $*$ denotes complex conjugation. Unless otherwise specified $q(x,t)$ or $q(\mathbf{x},t)$ is a complex valued function of the real variables $\mathbf{x}$ and t. There are also nonlocal matrix and vector extensions of many of the above equations. In this paper, we will show how Eqs. (3)–(13) arise from a rather simple but extremely important symmetry reductions of various AKNS scattering problems in one and multidimensions and show that they form a Hamiltonian integrable systems. For these equations, we provide few integrals of motions (conserved quantities) or indicate how an infinite number of them can be obtained and outline the solution strategy through the theory of IST. We then give a one soliton solution for a number of equations and discuss their properties.

In this paper, we do not discuss in detail vector or matrix extensions of the integrable nonlocal NLS equations, i.e., their PT-symmetric nonlocal versions, such as the equation obtained by replacing $[\mathbf{q}(x,t)\cdot\mathbf{q}(-x,-t)]$ in Eq. (3) by $[\mathbf{q}(x,t)\cdot\mathbf{q}^*(-x,t)]$ in which case the resulting PT symmetric multicomponent nonlocal NLS equation reads

$$i\mathbf{q}_t(x,t) = \mathbf{q}_{xx}(x,t) - 2\sigma[\mathbf{q}(x,t)\cdot\mathbf{q}^*(-x,t)]\mathbf{q}(x,t)\ . \tag{20}$$

As is the case in (3), here dot stands for the usual vector scalar product. We consider these equations to be direct extensions, though the IST is likely to contain novel aspects. In this regard, we note that direct and inverse scattering of the AKNS 2×2 and $n\times n$ systems have important applications in their own right.

The paper is organized as follows. In Section 2 we use the AKNS theory to derive various nonlocal reverse space-time and reverse time only NLS type equations in terms of two (complex or real) potentials: $q(x,t)$ and $r(x,t)$. In Section 3 we show how one can derive the nonlocal analogue of the derivative NLS equation and show that it is an integrable nonlocal system. We also give few conserved quantities. The derivation of nonlocal mKdV and sine-Gordon is given in Section 4. The extension of the reverse space-time and the reverse time nonlocal NLS equation to the multidimensional case, i.e., DS system is presented in Section 5. The partially PT symmetric and partially reverse space-time DS equations are obtained in Section 6. The derivation of the (1+1) and (2+1)-dimensional nonlocal in space and time analogue of the multiwave equations is presented in Sections 7 and 8, respectively. The discrete analogues for the above-mentioned nonlocal NLS equations are also derived in Section 9. For AKNS problems, the basic inverse scattering problem and reconstruction formula of the potential is developed in Section 10. The important symmetries of the associated eigenfunctions and scattering data together with soliton solutions is presented in Section 11. Finally, we conclude in Section 12 with an outlook for a future directions in the newly emerging field of integrable nonlocal equations including reverse space-time, reverse time, and PT symmetric nonlocal integrable systems.

2. Linear pair and compatibility conditions: Nonlocal NLS hierarchy

Our starting point is the AKNS scattering problem [6,29]

$$\mathbf{v}_x = \mathrm{X}\mathbf{v}, \tag{21}$$

where $\mathbf{v} = \mathbf{v}(x, t)$ is a two-component vector, i.e., $\mathbf{v}(x, t) = (v_1(x, t), v_2(x, t))^T$ and $q(x, t), r(x, t)$ are (in general) complex valued functions of the real variables x and t that vanish rapidly as $|x| \to \infty$ and k is a complex spectral parameter. The matrix X depends on the functions $q(x, t)$ and $r(x, t)$ as well as on the spectral parameter k

$$\mathsf{X} = \begin{pmatrix} -ik & q(x, t) \\ r(x, t) & ik \end{pmatrix}. \tag{22}$$

Associated with the scattering problem (21) is the time evolution equation of the eigenfunctions v_j, $j = 1, 2$ which is given by

$$\mathbf{v}_t = \mathsf{T}\mathbf{v}, \tag{23}$$

where

$$\mathsf{T} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A} \end{pmatrix}, \tag{24}$$

and the quantities A, B, and C are scalar functions of $q(x, t), r(x, t)$, and the spectral parameter k. Depending on the choice of these functions one finds an evolution equation for the potential functions $q(x, t)$ and $r(x, t)$ which, under a certain symmetry restriction, leads to a single evolution equation for either $q(x, t)$ or $r(x, t)$. In the case where the quantities A, B, and C are second-order polynomials in the isospectral parameter k with coefficients depending on $q(x, t), r(x, t)$, i.e.,

$$\mathsf{A} = 2ik^2 + iq(x, t)r(x, t), \tag{25}$$

$$\mathsf{B} = -2kq(x, t) - iq_x(x, t), \tag{26}$$

$$\mathsf{C} = -2kr(x, t) + ir_x(x, t), \tag{27}$$

the compatibility condition of system (21) and (23) leads to

$$iq_t(x, t) = q_{xx}(x, t) - 2r(x, t)q^2(x, t), \tag{28}$$

$$-ir_t(x, t) = r_{xx}(x, t) - 2q(x, t)r^2(x, t). \tag{29}$$

Under the symmetry reduction

$$r(x, t) = \sigma q(-x, -t), \sigma = \mp 1, \tag{30}$$

the system (28) and (29) are compatible and leads to the reverse space-time NLS equation (3), which for convenience we rewrite again:

$$iq_t(x, t) = q_{xx}(x, t) - 2\sigma q^2(x, t)q(-x, -t). \tag{31}$$

We remark that the symmetry reduction (30) is new and, because q is complex valued, is different from the symmetry

$$r(x,t) = \sigma q^*(-x,t). \tag{32}$$

The latter was found in [14] and leads to the PT symmetric nonlocal NLS Eq. (2). However, the new symmetry condition (30) gives rise to a new class of nonlocal (in both space and time) integrable evolution equations including a nonlocal NLS hierarchy. Equation (31) is another special and remarkably simple reduction of the more general q, r system mentioned above. For completeness, we give the compatible pair associated with Eq. (31):

$$\mathsf{X} = \begin{pmatrix} -ik & q(x,t) \\ \sigma q(-x,-t) & ik \end{pmatrix}, \tag{33}$$

$$\mathsf{T} = \begin{pmatrix} 2ik^2 + i\sigma q(x,t)q(-x,-t) & -2kq(x,t) - iq_x(x,t) \\ -2\sigma k q(-x,-t) - \sigma i q_x(-x,-t) & -2ik^2 - i\sigma q(x,t)q(-x,-t) \end{pmatrix}. \tag{34}$$

It is well known that the compatible pair (22)–(23) with (25)–(27) lead to an infinite number of conservation laws and conserved quantities cf. [6]. The first few conserved quantities associated with Eq. (31) are given by

$$\int_{\mathbb{R}} q(x,t)q(-x,-t)dx = \text{constant}, \tag{35}$$

$$\int_{\mathbb{R}} q_x(x,t)q(-x,-t)dx = \text{constant}, \tag{36}$$

$$\int_{\mathbb{R}} \left(\sigma q_x(x,t)q_x(-x,-t) + q^2(x,t)q^2(-x,-t)\right) dx = \text{constant}. \tag{37}$$

In the context of PT symmetric linear/nonlinear optics, the analogous quantity in Eq. (35) is referred to as the "quasipower." We also note that Eq. (31) is an integrable Hamiltonian system with Hamiltonian given by Eq. (37).

We also note that equations such as (31) are nonlocal in both space and time. Alone, it is not immediately clear how (31) is an evolution equation. However, with the symmetry reduction (30) we can consider (31) as arising as the unique solution associated with the evolution system (28)–(29) with initial conditions $r(x,t=0) = \sigma q^*(-x,t=0)$. All nonlocal in time equations in this paper can be considered in a similar way. Another interesting nonlocal symmetry reduction that system (28) and (29) admits is given by

$$r(x,t) = \sigma q(x,-t), \tag{38}$$

which, in turn, leads to the following new *reverse-time* NLS

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(x,-t). \tag{39}$$

Again, the condition (38) is new, remarkably simple, and has not been noticed in the literature and leads to a nonlocal in time NLS hierarchy. Furthermore, because this equation arises from the above AKNS scattering problem, it is an integrable Hamiltonian evolution equation that admits an infinite number of conservation laws/conserved quantities. The first few are listed below:

$$\int_{\mathbb{R}} q(x,t)q(x,-t)dx = \text{constant}, \tag{40}$$

$$\int_{\mathbb{R}} q(x,t)q_x(x,-t)dx = \text{constant}, \tag{41}$$

$$\int_{\mathbb{R}} \left(\sigma q_x(x,t)q_x(x,-t) + q^2(x,t)q^2(x,-t)\right) dx = \text{constant}. \tag{42}$$

The Lax pairs associated with Eq. (39) are thus given by

$$\mathsf{X} = \begin{pmatrix} -ik & q(x,t) \\ \sigma q(x,-t) & ik \end{pmatrix}, \tag{43}$$

$$\mathsf{T} = \begin{pmatrix} 2ik^2 + i\sigma q(x,t)q(x,-t) & -2kq(x,t) - iq_x(x,t) \\ -2\sigma kq(x,-t) \pm iq_x(x,-t) & -2ik^2 - i\sigma q(x,t)q(x,-t) \end{pmatrix}. \tag{44}$$

The extension to the matrix or vector (multicomponent) reverse space-time or reverse time only nonlocal NLS system can be carried out in a similar fashion. Indeed, if we start from the matrix generalization of the AKNS scattering problem then the compatibility condition generalizing the one given in system (28) and (29) would now read

$$i\mathbf{Q}_t(x,t) = \mathbf{Q}_{xx}(x,t) - 2\mathbf{Q}(x,t)\mathbf{R}(x,t)\mathbf{Q}(x,t), \tag{45}$$

$$-i\mathbf{R}_t(x,t) = \mathbf{R}_{xx}(x,t) - 2\mathbf{R}(x,t)\mathbf{Q}(x,t)\mathbf{R}(x,t), \tag{46}$$

where $\mathbf{Q}(x,t)$ is an $N \times M$ matrix; $\mathbf{R}(x,t)$ is an $M \times N$ matrix of the real variables x and t and super script T denotes matrix transpose (without complex conjugation). Under the symmetry reduction

$$\mathbf{R}(x,t) = \sigma\mathbf{Q}^T(-x,-t), \quad \sigma = \mp 1, \tag{47}$$

system (45) and (46) are compatible and this leads to the reverse space-time nonlocal matrix nonlinear Schrödinger equation

$$i\mathbf{Q}_t(x,t) = \mathbf{Q}_{xx}(x,t) - 2\sigma\mathbf{Q}(x,t)\mathbf{Q}^T(-x,-t)\mathbf{Q}(x,t). \tag{48}$$

In the special case where $\mathbf{Q}$ is either a column vector ($M = 1$) then Eq. (48) reduces to (3), i.e.,

$$i\mathbf{q}_t(x,t) = \mathbf{q}_{xx}(x,t) - 2\sigma[\mathbf{q}(x,t)\cdot\mathbf{q}(-x,-t)]\mathbf{q}(x,t), \tag{49}$$

where dot stands for the vector scalar product. As in the scalar case, we can generalize Eq. (39) to the matrix or vector multi component case. Indeed, we note that system (45) and (46) are compatible under the symmetry reduction

$$\mathbf{R}(x,t) = \sigma\mathbf{Q}^T(x,-t), \qquad \sigma = \mp 1, \tag{50}$$

which in turn gives rise to the following nonlocal in time only matrix nonlinear Schrödinger equation

$$i\mathbf{Q}_t(x,t) = \mathbf{Q}_{xx}(x,t) - 2\sigma\mathbf{Q}(x,t)\mathbf{Q}^T(x,-t)\mathbf{Q}(x,t). \tag{51}$$

To obtain the multicomponent analogue of Eq. (51) we restrict the matrix $\mathbf{Q}$ to a column vector ($N = 1$) giving rise to the following nonlocal evolution equation:

$$i\mathbf{q}_t(x,t) = \mathbf{q}_{xx}(x,t) - 2\sigma[\mathbf{q}(x,t)\cdot\mathbf{q}(x,-t)]\mathbf{q}(x,t). \tag{52}$$

3. Reverse space-time nonlocal coupled NLS-derivative NLS equation

In this section, we derive the space-time nonlocal coupled NLS-derivative NLS equation that includes the reverse space-time nonlocal NLS (as well as the reverse space-time nonlocal derivative NLS) equations as special cases. To do so we consider a generalization to the AKNS scattering problem (21) with

$$\mathsf{X} = \begin{pmatrix} -f(k) & g(k)q(x,t) \\ g(k)r(x,t) & f(k) \end{pmatrix}, \tag{53}$$

where $f(k) = i\alpha k^2 - \sqrt{2\beta}k$ and $g(k) = \alpha k + i\sqrt{\beta/2}$ are functions of the complex spectral parameter k and α, β are real constants. The time evolution of the eigenfunctions $\mathbf{v}(x,t)$ is governed by Eqs. (23) and (24) where functions A, B, and C are now fourth-order polynomials in the isospectral parameter k (see [13]). The compatibility condition of system (53) and (23) gives the coupled q, r system

$$q_t(x,t) = iq_{xx}(x,t) + \alpha\left(r(x,t)q^2(x,t)\right)_x + i\beta r(x,t)q^2(x,t), \tag{54}$$

$$-r_t(x,t) = ir_{xx}(x,t) - \alpha\left(r^2(x,t)q(x,t)\right)_x + i\beta r^2(x,t)q(x,t). \tag{55}$$

Under the symmetry reduction (30) the system (54) and (55) are compatible and leads to the reverse space-time nonlocal coupled NLS-derivative NLS equation:

$$q_t(x,t) = iq_{xx}(x,t) + \alpha\sigma\left(q(-x,-t)q^2(x,t)\right)_x + i\beta\sigma q(-x,-t)q^2(x,t). \tag{56}$$

In the special case where $\alpha = 0$ and $\beta = 2$ we recover Eq. (31). On the other hand, if we choose $\alpha = 1$ and $\beta = 0$ then we find the reverse space-time nonlocal version of the "classical" derivative NLS equation:

$$iq_t(x,t) = -q_{xx}(x,t) + i\sigma\left(q(-x,-t)q^2(x,t)\right)_x. \tag{57}$$

The linear Lax pairs associated with Eq. (57) are given by

$$\mathsf{X} = \begin{pmatrix} -ik^2 & kq(x,t) \\ k\sigma q(-x,-t) & ik^2 \end{pmatrix}, \tag{58}$$

$$\mathsf{T} = \begin{pmatrix} \mathsf{A}^{nonloc}_{dNLS} & \mathsf{B}^{nonloc}_{dNLS} \\ \mathsf{C}^{nonloc}_{dNLS} & -\mathsf{A}^{nonloc}_{dNLS} \end{pmatrix}, \tag{59}$$

where

$$\mathsf{A}^{nonloc}_{dNLS} = -2ik^4 - i\sigma q(-x,-t)q(x,t)k^2, \tag{60}$$

$$\mathsf{B}^{nonloc}_{dNLS} = 2q(x,t)k^3 + (iq_x(x,t) + \sigma q(-x,-t)q^2(x,t))k, \tag{61}$$

$$\mathsf{C}^{nonloc}_{dNLS} = 2\sigma kq(-x,-t)k^3 + (i\sigma q_x(-x,-t) + q^2(-x,-t)q(x,t))k. \tag{62}$$

In [30] it was shown that the general q, r system (54) and (55) for $\alpha = 1$ and $\beta = 0$ is integrable and admits infinitely many conservation laws. Because the new nonlocal equation (57) comes out of a new symmetry reduction it is also an infinite dimensional integrable Hamiltonian system. The first two conserved quantities associated with Eq. (57) are

$$\int_{\mathbb{R}} q(x,t)q(-x,-t)dx = \text{constant}, \tag{63}$$

$$\int_{\mathbb{R}} q(x,t)\left[\frac{i}{2}q^2(-x,-t)q(x,t) - \sigma q_x(-x,-t)\right]dx = \text{constant}. \tag{64}$$

Another interesting nonlocal in both space and time integrable evolution equation can be obtained from the scattering problem (53) if one chooses the functional dependence of f and g on k to be linear, i.e., $f(k) = g(k) =$

k with suitable functions A, B, and C (see [13, 31]). Following the same procedure as above, the compatibility condition gives rise to the following system of q, r equations:

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left[\frac{q_x(x,t)}{(1-r(x,t)q(x,t))^{3/2}}\right] = 0, \tag{65}$$

$$\frac{\partial r(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left[\frac{r_x(x,t)}{(1-r(x,t)q(x,t))^{3/2}}\right] = 0. \tag{66}$$

Now, under the symmetry reduction (30), i.e., $r(x,t) = \sigma q(-x,-t), \sigma = \mp 1$, Eqs. (65) and (66) are compatible and leads to the reverse space-time nonlocal "loop soliton" equation

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\left[\frac{q_x(x,t)}{(1-\sigma q(x,t)q(-x,-t))^{3/2}}\right] = 0, \tag{67}$$

with $\sigma = \mp 1$. The conservation laws for this "loop soliton" system can be obtained by standard methods; cf. [32, 33]

4. Complex and real reverse space-time nonlocal mKdV and sine-Gordon Equations

Returning to the 2×2 Lax pair given by Eqs. (21)–(24) we can find other integrable nonlocal equations depending on the functional form of A, B, and C on the spectral parameter k. In the following few sections, we will derive the space-time nonlocal versions of the "classical" mKdV and sine-Gordon equations and provide the IST formulation as well as one soliton solution. Contrary to the PT symmetric nonlocal NLS case where the one soliton solution can develop a singularity in fine time [14, 22], here the reverse space-time nonlocal mKdV soliton can be generically regular and does not develop a singularity.

4.1. The complex reverse space-time nonlocal mKdV

If we take

$$\mathsf{A}_3 = -4ik^3 - 2iq(x,t)r(x,t)k + r(x,t)q_x(x,t) - q(x,t)r_x(x,t),$$
$$\mathsf{B}_3 = 4k^2 q(x,t) + 2iq_x(x,t)k + 2q^2(x,t)r(x,t) - q_{xx}(x,t),$$
$$\mathsf{C}_3 = 4k^2 r(x,t) - 2ir_x(x,t)k + 2q(x,t)r^2(x,t) - r_{xx}(x,t),$$

the compatibility condition of system (21) and (23) yields

$$q_t(x,t) + q_{xxx}(x,t) - 6q(x,t)r(x,t)q_x(x,t) = 0, \tag{68}$$

$$r_t(x,t) + r_{xxx}(x,t) - 6q(x,t)r(x,t)r_x(x,t) = 0. \tag{69}$$

Under the symmetry reduction

$$r(x,t) = \sigma q^*(-x,-t), \sigma = \mp 1, \tag{70}$$

the system (68) and (69) are compatible and leads to the complex reverse space-time nonlocal complex mKdV equation

$$q_t(x,t) + q_{xxx}(x,t) - 6\sigma q(x,t)q^*(-x,-t)q_x(x,t) = 0, \tag{71}$$

where again $*$ denotes complex conjugation and $q(x,t)$ is a complex valued function of the real variables x and t. On the other hand, using the symmetry (30) yields the real reverse space-time equation

$$q_t(x,t) + q_{xxx}(x,t) - 6\sigma q(x,t)q(-x,-t)q_x(x,t) = 0, \tag{72}$$

which for real initial conditions is the real nonlocal mKdV equation. We also point out that when $q(-x,-t) = q(x,t)$ the nonlocal mKdV equation reduces to its classical (local) counterpart. Furthermore, when using the symmetry reduction $r(x,t) = \sigma q(-x,-t)$ for the NLS or mKdV case, one need not specify whether q is real or complex valued. However, if one further restricts q to be real then additional symmetry conditions on the underlying eigenfunctions and scattering data are required, beyond those that come out of the symmetry reduction $r(x,t) = \sigma q^*(-x,-t)$. The compatible pair associated with Eq. (71) now is

$$\mathbf{v}_x = \begin{pmatrix} -ik & q(x,t) \\ \sigma q^*(-x,-t) & ik \end{pmatrix} \mathbf{v}, \tag{73}$$

$$\mathbf{v}_t = \begin{pmatrix} \mathsf{A}_{3,nonloc} & \mathsf{B}_{3,nonloc} \\ \mathsf{C}_{3,nonloc} & -\mathsf{A}_{3,nonloc} \end{pmatrix} \mathbf{v}, \tag{74}$$

where

$$\begin{aligned} \mathsf{A}_{3,nonloc} = &-4ik^3 - 2i\sigma q(x,t)q^*(-x,-t)k \\ &+\sigma q^*(-x,-t)q_x(x,t) + \sigma q(x,t)q_x^*(-x,-t), \end{aligned} \tag{75}$$

$$\begin{aligned} \mathsf{B}_{3,nonloc} = &\, 4k^2 q(x,t) + 2iq_x(x,t)k \\ &+2\sigma q^2(x,t)q^*(-x,-t) - q_{xx}(x,t), \end{aligned} \tag{76}$$

$$\begin{aligned} \mathsf{C}_{3,nonloc} = &\, 4k^2\sigma q^*(-x,-t) + 2i\sigma q_x^*(-x,-t)k \\ &+2q(x,t)q^{*2}(-x,-t) - \sigma q_{xx}^*(-x,-t). \end{aligned} \tag{77}$$

4.2. The reverse space-time nonlocal sine-Gordon equation

If on the other hand, one makes the ansatz $\mathsf{A} = \mathsf{A}_1/k, \mathsf{B} = \mathsf{B}_1/k$, and $\mathsf{C} = \mathsf{C}_1/k$ then after some algebra the compatibility condition $v_{jxt} = v_{jtx}, j = 1, 2$ with k being the time independent isospectral parameter leads to

$$q_{xt}(x,t) + 2s(x,t)q(x,t) = 0, \tag{78}$$

$$r_{xt}(x,t) + 2s(x,t)r(x,t) = 0, \tag{79}$$

$$s_x(x,t) + (q(x,t)r(x,t))_t = 0, \tag{80}$$

where we have defined $\mathsf{A}_1 = -is/2$. Also for completeness: $\mathsf{B}_1 = q_t/(2i)$, $\mathsf{C}_1 = -r_t/(2i)$. Under the symmetry condition

$$r(x,t) = -q(-x,-t), \tag{81}$$

with $q \in \mathbb{R}$ the system of equations (78)–(80) are compatible and give rise to the real nonlocal sine-Gordon (sG) equation

$$q_{xt}(x,t) + 2s(x,t)q(x,t) = 0, \quad s(-x,-t) = s(x,t). \tag{82}$$

We also fix the boundary condition of s as $x \to \infty$ consistent with the classical sine-Gordon equation to be

$$s(x,t) = s(-\infty) - \int_{-\infty}^{x} (q(x,t)q(-x,-t))_t(x',t)dx', s(-\infty) = i/4. \tag{83}$$

We note that we could have also generated a complex form of the (sG) equation following the previous discussion. However, for simplicity, here we only give the real nonlocal (SG) equation.

4.3. Overview

In summary, system (28) and (29) admits six symmetry reductions. The first four of which give rise to an integrable nonlocal NLS-type equation and the last two of which yield a nonlocal integrable mKdV-type evolution equation (below $\sigma = \mp 1$):

1. Standard AKNS symmetry:

   $$r(x,t) = \sigma q^*(x,t),$$

   which has been known in the literature for more than four decades [12]; the paradigm is the NLS equation (1);

2. Reverse time AKNS symmetry

$$r(x,t) = \sigma q(x,-t),$$

see the NLS-type equation (3);

3. PT preserving symmetry

$$r(x,t) = \sigma q^*(-x,t),$$

which was found in 2013 [14]; see the NLS-type equation (2);

4. Reverse space-time symmetry

$$r(x,t) = \sigma q(-x,-t), q \in \mathbb{C},$$

see the NLS-type equation (3);

5. PT reverse time symmetry

$$r(x,t) = \sigma q^*(-x,-t),$$

see the complex mKdV-type equation (3);

6. Real reverse space-time symmetry

$$r(x,t) = \sigma q(-x,-t), q \in \mathbb{R},$$

see the real mKdV-type equation (3).

In Section 11, we find soliton solutions of nonlocal NLS, mKdV and sine-Gordon–type equations with these symmetries.

5. Reverse space-time and reverse time nonlocal DS system

The integrable two spatial dimensional extension of the NLS equation was obtained from a 2×2 compatible linear pair in [34]. The IST was carried out later—cf. [13]. The spatial part of the linear pair generalizes the operator X in (21) and (22) where the eigenvalue k is replaced by an operator in the transverse spatial variable y. This new operator still contains the potentials q, r which now depend on x, y, and t. The general DS (q, r) system is given by

$$iq_t(\mathbf{x},t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t) + q_{yy}(\mathbf{x},t)\right] + q^2(\mathbf{x},t)r(\mathbf{x},t) = \phi(\mathbf{x},t)q(\mathbf{x},t), \quad (84)$$

$$-ir_t(\mathbf{x},t) + \frac{1}{2}\left[\gamma^2 r_{xx}(\mathbf{x},t) + r_{yy}(\mathbf{x},t)\right] + r^2(\mathbf{x},t)q(\mathbf{x},t) = \phi(\mathbf{x},t)r(\mathbf{x},t), \quad (85)$$

$$\phi_{xx}(\mathbf{x},t) - \gamma^2 \phi_{yy}(\mathbf{x},t) = 2\left[q(\mathbf{x},t)r(\mathbf{x},t)\right]_{xx}, \quad (86)$$

where $\gamma^2 = \pm 1$ and $\mathbf{x} = (x, y)$ is the transverse plane. In [34] it was shown that the system of equations (84) and (85) are consistent under the symmetry reduction $r(\mathbf{x}, t) = \sigma q^*(\mathbf{x}, t)$ and leads to the "classical" DS equation and in [24] a PT symmetric reduction in the form $r(\mathbf{x}, t) = \sigma q^*(-\mathbf{x}, t)$ was also reported. In this paper, we identify two new nonlocal symmetry reductions to the above DS system: $r(\mathbf{x}, t) = \sigma q(-\mathbf{x}, -t)$ and $r(\mathbf{x}, t) = \sigma q(\mathbf{x}, -t)$ each of which leads to a new DS system.

5.1. Reverse space-time nonlocal DS equation

Under the symmetry reduction

$$r(\mathbf{x}, t) = \sigma q(-\mathbf{x}, -t), \tag{87}$$

it can be shown that system (84) and (85) are compatible and lead to the reverse space-time nonlocal DS equation (12) which, for the convenience of the reader we rewrite again:

$$iq_t(\mathbf{x}, t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x}, t) + q_{yy}(\mathbf{x}, t)\right] + \sigma q^2(\mathbf{x}, t)q(-\mathbf{x}, -t) = \phi(\mathbf{x}, t)q(\mathbf{x}, t), \tag{88}$$

$$\phi_{xx}(\mathbf{x}, t) - \gamma^2\phi_{yy}(\mathbf{x}, t) = 2\sigma\left[q(\mathbf{x}, t)q(-\mathbf{x}, -t)\right]_{xx}. \tag{89}$$

Note that from Eq. (89) it follows that the potential ϕ has a solution that satisfies the relation $\phi(-\mathbf{x}, -t) = \phi(\mathbf{x}, t)$. The solution ϕ can, in principle, have boundary conditions that do not allow $\phi(-\mathbf{x}, -t) = \phi(\mathbf{x}, t)$. For the decaying infinite space problem we are considering here, one can expect the symmetry relation for $\phi(\mathbf{x}, t)$ to hold. The elliptic case in the ϕ equation is easier than the hyperbolic one. In general, to prove $\phi(-\mathbf{x}, -t) = \phi(\mathbf{x}, t)$ one need to study the Greens function and see if this symmetry reduction holds. For the two-dimensional elliptic case with $\gamma^2 = -1$ this condition appears to be true. Thus, the existence of the symmetry property for $\phi(\mathbf{x}, t)$ is *necessary* for the (q, r) DS system to be compatible. Any solution $\phi(\mathbf{x}, t)$ that breaks the symmetry $\phi(-\mathbf{x}, -t) = \phi(\mathbf{x}, t)$, would force the (q, r) system to become inconsistent. As such, the proposed new nonlocal DS equations are valid so long $\phi(\mathbf{x}, t)$ satisfy the necessary underlying symmetry induced from the nonlocal AKNS symmetry reduction.

5.2. Reverse time nonlocal DS equation

Another interesting and new symmetry reduction which was not noticed in the literature so far is the time only nonlocal reduction given by

$$r(\mathbf{x}, t) = \sigma q(\mathbf{x}, -t), \qquad \sigma = \mp 1. \tag{90}$$

With this symmetry condition, system (84) and (85) are consistent and give rise to the following reverse time-only nonlocal DS system of equation

$$iq_t(\mathbf{x},t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t) + q_{yy}(\mathbf{x},t)\right] + \sigma q^2(\mathbf{x},t)q(\mathbf{x},-t) = \phi(\mathbf{x},t)q(\mathbf{x},t), \tag{91}$$

$$\phi_{xx}(\mathbf{x},t) - \gamma^2\phi_{yy}(\mathbf{x},t) = 2\sigma\left[q(\mathbf{x},t)q(\mathbf{x},-t)\right]_{xx}. \tag{92}$$

Note that from Eqs. (91) and (92) it follows that the potential ϕ has a solution that satisfies $\phi(\mathbf{x},t) = \phi(\mathbf{x},-t)$. In this paper, we will not go into further detail regarding the integrability properties of the above systems nor will we construct soliton solutions or an inverse scattering theory. This will be left for future work.

6. Fully PT symmetric, partially PT symmetric, and partial reverse space-time nonlocal DS system

In this section, we show that the (DS) system (84) and (85) admit yet other types of symmetry reductions. These new symmetry reductions fall into three distinct categories: (i) full PT symmetry, (ii) partial PT symmetry, and (iii) partial reverse space-time symmetry. Generally speaking, a linear or nonlinear PDE is said to be PT symmetric if it is invariant under the combined action of the (not necessarily linear) PT operator. In $(1+1)$ dimensions, this amounts to invariance under the joint transformation $x \to -x, t \to -t$ and complex conjugation. The situation for the $(2+1)$ case is more rich. Here, one can talk about two different types of PT symmetries: full and partial. If we denote by $\mathbf{x} \equiv (x, y)$, then a linear or nonlinear PDE in $(2+1)$ dimensions is said to be fully PT symmetric if it is invariant under the combined transformation $\mathbf{x} \to -\mathbf{x}$ (parity operator P), $t \to -t$ plus complex conjugation (T operator). Note that the space reflection is performed in both space coordinates. On the other hand, a linear or nonlinear PDE in $(2+1)$ dimensions is said to be partially PT symmetric if it is invariant under the combined transformation $(x, y) \to (-x, y)$ or $(x, y) \to (x, -y)$, $t \to -t$ plus complex conjugation. Partially PT symmetric optical potentials have been studied in [23] and shown that such potentials exhibit pure real spectra and can support (in the presence of cubic type nonlinearity) continuous families of solitons. Below, we use these new symmetry reductions to derive the corresponding DS–like equations.

6.1. Partially PT symmetric nonlocal DS equation

Under the symmetry reduction

$$r(x, y, t) = \sigma q^*(-x, y, t), \tag{93}$$

it can be shown that system (84) and (85) are compatible and lead to the partially PT symmetric nonlocal DS equation (14) which, for the convenience of the reader we rewrite again:

$$i q_t(x, y, t) + \frac{1}{2}\left[\gamma^2 q_{xx}(x, y, t) + q_{yy}(x, y, t)\right] \\ + \sigma q^2(x, y, t) q^*(-x, y, t) = \phi(x, y, t) q(x, y, t), \tag{94}$$

$$\phi_{xx}(x, y, t) - \gamma^2 \phi_{yy}(x, y, t) = 2\sigma \left[q(x, y, t) q^*(-x, y, t)\right]_{xx}. \tag{95}$$

Note that from Eq. (95) it follows that the potential ϕ has a solution that satisfies the relation $\phi(x, y, t) = \phi^*(-x, y, t)$, in other words, the potential satisfies the partial PT symmetry requirement. Note, here and below we could have also considered the partial PT reduction:

$$r(x, y, t) = \sigma q^*(x, -y, t), \tag{96}$$

which would lead to another DS-type equation.

6.2. Partial reverse space-time nonlocal DS equation

Another new symmetry reduction, which was not noticed in the literature so far is the partially reverse space-time nonlocal reduction given by

$$r(x, y, t) = \sigma q(-x, y, -t), \qquad \sigma = \mp 1. \tag{97}$$

With this symmetry condition, system (84) and (85) are consistent and give rise to the following partially reverse space-time nonlocal DS system of equation

$$i q_t(x, y, t) + \frac{1}{2}\left[\gamma^2 q_{xx}(x, y, t) + q_{yy}(x, y, t)\right] \\ + \sigma q^2(x, y, t) q(-x, y, -t) = \phi(x, y, t) q(x, y, t), \tag{98}$$

$$\phi_{xx}(x, y, t) - \gamma^2 \phi_{yy}(x, y, t) = 2\sigma \left[q(x, y, t) q(-x, y, -t)\right]_{xx}. \tag{99}$$

Note that from Eqs. (98) and (99) it follows that the potential ϕ has a solution that satisfies $\phi(x, y, t) = \phi(-x, y, -t)$. Again, one can consider the partial reverse space-time reduction:

$$r(x, y, t) = \sigma q(x, -y, -t), \tag{100}$$

and obtain the corresponding DS equation. In this paper, we will not go into further detail regarding the integrability properties of the above systems nor will we construct soliton solutions or an inverse scattering theory. This will be left for future work. We close this section by mentioning that the fully PT symmetric nonlocal DS equation was obtained by Fokas in [24]. Indeed, under the symmetry condition $r(\mathbf{x}, t) = \sigma q^*(-\mathbf{x}, t)$ the system (84) and (85) are again compatible and lead to the following PT symmetric nonlocal DS equation [24]:

$$iq_t(\mathbf{x}, t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x}, t) + q_{yy}(\mathbf{x}, t)\right] + \sigma q^2(\mathbf{x}, t)q^*(-\mathbf{x}, t) = \phi(\mathbf{x}, t)q(\mathbf{x}, t) = 0, \tag{101}$$

$$\phi_{xx}(\mathbf{x}, t) - \gamma^2\phi_{yy}(\mathbf{x}, t) = 2\sigma\left[q(\mathbf{x}, t)q^*(-\mathbf{x}, t)\right]_{xx}, \tag{102}$$

with the potential $\phi(\mathbf{x}, t)$ satisfying the PT symmetry condition: $\phi^*(-\mathbf{x}, t) = \phi(\mathbf{x}, t)$. In summary, like the integrable NLS-type equations, the DS system (84), (85) and (86) admit six different symmetry reductions that we list below:

1. Classical $r(\mathbf{x}, t) = \sigma q^*(\mathbf{x}, t)$ observed in [35],
2. Fully PT symmetric: $r(\mathbf{x}, t) = \sigma q^*(-\mathbf{x}, t)$ reported in [24],
3. Partially PT symmetric: $r(x, y, t) = \sigma q^*(-x, y, t)$ or $r(x, y, t) = \sigma q^*(x, -y, t)$ found in this paper,
4. Reverse space-time symmetry $r(\mathbf{x}, t) = \sigma q(-\mathbf{x}, -t)$ found in this paper,
5. Partial reverse space-time symmetry $r(x, y, t) = \sigma q(-x, y, -t)$ or $r(x, y, t) = \sigma q(x, -y, -t)$ found in this paper,
6. Reverse time symmetry $r(\mathbf{x}, t) = \sigma q(\mathbf{x}, -t)$ found in this paper.

It would be interesting for future research direction to study the solutions and possible wave collapse properties (or lack of) for each of the reported new reductions.

7. (1+1)-dimensional reverse space-time nonlocal multiwave and three-wave equations

In this section, we derive the reverse space-time and reverse time nonlocal multiwave equation and its physically important reduction to three wave equations. The idea is to generalize the 2×2 AKNS scattering problem (21) and its associated time evolution (23) to an $n \times n$ matrix form and obtain, after following similar compatibility procedure, the corresponding multi interacting nonlinear, nonlocal (in space and time) wave equation.

A physically relevant reduction of the more general case, i.e., three-wave equation will be also derived. Our approach follows that given by Ablowitz and Haberman [35]. An $n \times n$ generalization of the scattering problem (21) and (22) is given by

$$\mathbf{v}_x = ik\mathsf{D}\mathbf{v} + \mathsf{N}\mathbf{v}, \tag{103}$$

where $\mathbf{v}$ is a column vector of length n, i.e., $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T$ where as before, T denotes matrix transpose. Furthermore, D and N are $n \times n$ matrices with D being a diagonal constant matrix, i.e., $\mathsf{D} \equiv \mathrm{diag}(d_1, d_2, \cdots, d_n)$, $d_n > d_{n-1} \cdots > d_1$, and N has zero entries on the main diagonal; i.e., in matrix element form $N_{\ell,\ell} = 0$. The time evolution associated with (103) is given by

$$\mathbf{v}_t = \mathsf{Q}\mathbf{v}, \tag{104}$$

with Q being an $n \times n$ matrix which depends on the components of the "potential" matrix N and the assumed time-independent spectral parameter k. As in the 2×2 case, the compatibility condition $\mathbf{v}_{xt} = \mathbf{v}_{tx}$ yields the matrix equation

$$\mathsf{Q}_x - \mathsf{N}_t = ik[\mathsf{D}, \mathsf{Q}] + [\mathsf{N}, \mathsf{Q}], \tag{105}$$

where $[\mathsf{A}, \mathsf{B}] \equiv \mathsf{AB} - \mathsf{BA}$. If one now expands the matrix Q in a first-order polynomial in the spectral parameter k, $\mathsf{Q} = \mathsf{Q}_0 + k\mathsf{Q}_1$ then, after some algebra, one finds $\mathsf{Q}_{1\ell j} \equiv q_\ell \delta_{\ell j}$ and $\mathsf{Q}_{0\ell j} = a_{\ell j}\mathsf{N}_{\ell j}$ and $a_{\ell j} = -\frac{i(q_\ell - q_j)}{(d_\ell - d_j)} = a_{j\ell}$. We want $a_{\ell j} \in \mathbb{R}$ hence q_j, $j = 1, \cdots n$ are purely imaginary. The time evolution of the matrix elements $\mathsf{N}_{\ell j}$, $\ell, j = 1, 2, \cdots, n$ is found at order k^0 and given by

$$\mathsf{N}_{\ell j,t}(x,t) - a_{\ell j}\mathsf{N}_{\ell j,x}(x,t) = \sum_{m=1}^{n}(a_{\ell m} - a_{mj})\mathsf{N}_{\ell m}(x,t)\mathsf{N}_{mj}(x,t). \tag{106}$$

Equation (106) was derived in [35] and governs the time evolution of generic "potential" matrix elements $\mathsf{N}_{\ell j}$.

7.1. *Classical multiwave reduction:* $\mathsf{N}_{\ell j}(x,t) = \sigma_{\ell j}\mathsf{N}^*_{j\ell}(x,t)$

In [35] it was shown that the system of equations (106) admits the following symmetry reduction

$$\mathsf{N}_{\ell j}(x,t) = \sigma_{\ell j}\mathsf{N}^*_{j\ell}(x,t), \tag{107}$$

where $\sigma_{\ell j}$ are constants satisfying

$$\sigma^2_{\ell j} = 1, \sigma_{\ell m}\sigma_{mj} = -\sigma_{\ell j},$$

for all $m, \ell, j = 1, 2, \cdots, n$ and real $a_{\ell m}$. That is to say, $\mathsf{N}_{\ell j}(x,t)$ and $\mathsf{N}^*_{j\ell}(x,t)$ satisfy the same equation (106) (and its complex conjugate) thus reducing the number of equation by half; there are $n(n-1)/2$ interacting nonlinear wave equations. These equations have an infinite number of conservation laws [36]

7.2. Classical three-wave interaction equations

The physically relevant and important local three wave interaction system is next derived. In this case $n = 3$ and the "nonlinear" matrix N is assumed to have the generic form (note that $\mathsf{N}_{jj} = 0$, $j = 1, 2, 3$)

$$\mathsf{N}(x,t) = \begin{pmatrix} 0 & \mathsf{N}_{12}(x,t) & \mathsf{N}_{13}(x,t) \\ \mathsf{N}_{21}(x,t) & 0 & \mathsf{N}_{23}(x,t) \\ \mathsf{N}_{31}(x,t) & \mathsf{N}_{32}(x,t) & 0 \end{pmatrix}. \tag{108}$$

With the symmetry (107), one can reduce the number of independent variables in (108) and write

$$\mathsf{N}(x,t) = \begin{pmatrix} 0 & \mathsf{N}_{12}(x,t) & \mathsf{N}_{13}(x,t) \\ \sigma_1\mathsf{N}^*_{12}(x,t) & 0 & \mathsf{N}_{23}(x,t) \\ \sigma_2\mathsf{N}^*_{13}(x,t) & \sigma_3\mathsf{N}^*_{23}(x,t) & 0 \end{pmatrix}, \tag{109}$$

where

$$\frac{\sigma_1\sigma_1}{\sigma_2} = 1, \sigma_j = \pm 1, j = 1, 2, 3.$$

Thus, the number of nonlinear equations is reduced from 6 to 3. Next, we consider the following transformation of variables,

$$\mathsf{N}_{12}(x,t) = -i\frac{Q_3(x,t)}{\sqrt{\beta_{13}\beta_{23}}},$$

$$\mathsf{N}_{31}(x,t) = -i\frac{Q_2(x,t)}{\sqrt{\beta_{12}\beta_{23}}},$$

$$\mathsf{N}_{23}(x,t) = i\frac{Q_1(x,t)}{\sqrt{\beta_{12}\beta_{13}}},$$

$$\mathsf{N}_{13}(x,t) = -\gamma_1\gamma_3\mathsf{N}^*_{31}(x,t),$$

$$\mathsf{N}_{32}(x,t) = \gamma_3\gamma_2\mathsf{N}^*_{23}(x,t),$$

$$\mathsf{N}_{21}(x,t) = \gamma_1\gamma_2\mathsf{N}^*_{12}(x,t),$$

where

$$\beta_{lj} := d_l - d_j = -c_l + c_j, \Rightarrow d_j = -c_j \Rightarrow c_3 > c_2 > c_1$$
$$\gamma_j = -i\frac{c_1c_2c_3}{c_j}.$$

This results in the classical (local) three wave interaction equations

$$
\begin{aligned}
Q_{1,t}(x,t) + c_1 Q_{1,x}(x,t) &= i\gamma_1 Q_2^*(x,t) Q_3^*(x,t), \\
Q_{2,t}(x,t) + c_2 Q_{2,x}(x,t) &= i\gamma_2 Q_1^*(x,t) Q_3^*(x,t), \\
Q_{3,t}(x,t) + c_3 Q_{3,x}(x,t) &= i\gamma_3 Q_1^*(x,t) Q_2^*(x,t),
\end{aligned} \tag{110}
$$

where

$$
c_3 > c_2 > c_1, \gamma_1\gamma_2\gamma_3 = -1, \gamma_j = \pm 1, j = 1, 2, 3.
$$

From these equations, we can derive the conserved quantities

$$
\begin{aligned}
\gamma_1 \int_{-\infty}^{\infty} |Q_1(x,t)|^2 dx - \gamma_2 \int_{-\infty}^{\infty} |Q_2(x,t)|^2 dx &= \text{constant}, \\
\gamma_2 \int_{-\infty}^{\infty} |Q_2(x,t)|^2 dx - \gamma_3 \int_{-\infty}^{\infty} |Q_3(x,t)|^2 dx &= \text{constant}, \\
\gamma_1 \int_{-\infty}^{\infty} |Q_1(x,t)|^2 dx - \gamma_3 \int_{-\infty}^{\infty} |Q_3(x,t)|^2 dx &= \text{constant}.
\end{aligned} \tag{111}
$$

Positive definite energy occurs when we take two γ_j's of different sign. This results in the "decay instability" case. If all three $\gamma_j = -1$ then the above does not lead to a positive definite energy—this is the "explosive instability" case. Next we show that the system (106) admits *new space-time* nonlocal symmetry reductions leading to nonlocal multiwave equations. We will discuss two reductions.

7.3. The complex reverse space-time multiwave reduction: $\mathsf{N}_{\ell j}(x,t) = \sigma_{\ell j}\mathsf{N}^*_{j\ell}(-x,-t)$

In this section we show that the system of multi-interacting waves admits a new nonlocal symmetry reduction. Later, we derive a simple model of a nonlocal three-wave equation. We substitute in Eq. (106) the new symmetry relation

$$
\mathsf{N}_{\ell j}(x,t) = \sigma_{\ell j}\mathsf{N}^*_{j\ell}(-x,-t), \tag{112}
$$

and call $x' = -x, t' = -t$ and find:

$$
-(N^*_{j\ell,t'} - a_{lj} N^*_{j\ell,x'})(x',t') = \sum_{m=1}^{n} (a_{\ell m} - a_{mj}) \frac{\sigma_{\ell m}\sigma_{mj}}{\sigma_{\ell j}} N^*_{m\ell}(x',t') N^*_{jm}(x',t'). \tag{113}
$$

Under the condition

$$
\frac{\sigma_{\ell m}\sigma_{mj}}{\sigma_{\ell j}} = +1,
$$

Eq. (113) agrees with the complex conjugate of Eq. (106) with interchanged indices.

7.4. Complex reverse space-time three-wave equations

With the symmetry reduction $\mathsf{N}_{21} = \sigma_1 \mathsf{N}^*_{12}(-x,-t)$, $\mathsf{N}_{31} = \sigma_2 \mathsf{N}^*_{13}(-x,-t)$ and $\mathsf{N}_{32} = \sigma_3 \mathsf{N}^*_{23}(-x,-t)$ and assuming that a_{lj} are real, where σ_1, σ_2 and σ_3 are chosen as real numbers with

$$\frac{\sigma_1\sigma_3}{\sigma_2} = 1, \sigma_j = \pm 1, j = 1, 2, 3, \tag{114}$$

Eq. (106) may be put into a set of nonlocal three-wave interaction equations by a suitable scaling of variables. For example, we find the system

$$\begin{aligned} Q_{1,t}(x,t) + c_1 Q_{1,x}(x,t) &= \sigma_3 Q^*_2(-x,-t) Q^*_3(-x,-t),\\ Q_{2,t}(x,t) + c_2 Q_{2,x}(x,t) &= -\sigma_2 Q^*_1(-x,-t) Q^*_3(-x,-t),\\ Q_{3,t}(x,t) + c_3 Q_{3,x}(x,t) &= \sigma_1 Q^*_1(-x,-t) Q^*_2(-x,-t), \end{aligned} \tag{115}$$

if we take

$$\mathsf{N}_{12}(x,t) = -\frac{Q_3(x,t)}{\sqrt{\beta_{13}\beta_{23}}},$$

$$\mathsf{N}_{31}(x,t) = -\frac{Q_2(x,t)}{\sqrt{\beta_{12}\beta_{23}}},$$

$$\mathsf{N}_{23}(x,t) = -\frac{Q_1(x,t)}{\sqrt{\beta_{12}\beta_{13}}},$$

$$\mathsf{N}_{13}(x,t) = \sigma_2 \mathsf{N}^*_{31}(-x,-t),$$

$$\mathsf{N}_{32}(x,t) = \sigma_3 \mathsf{N}^*_{23}(-x,-t),$$

$$\mathsf{N}_{21}(x,t) = \sigma_1 \mathsf{N}^*_{12}(-x,-t),$$

where

$$\begin{aligned} &\beta_{lj} := d_l - d_j = -c_l + c_j, \Rightarrow d_j = -c_j \Rightarrow d_1 = -c_1,\\ &d_2 = -c_2, d_3 = -c_3\\ &q_1 = -ic_2c_3,\ q_2 = -ic_1c_3,\ q_3 = -ic_1c_2,\\ &a_{12} = -c_3,\ a_{13} = -c_2,\ a_{23} = -c_1, c_3 > c_2 > c_1. \end{aligned}$$

Directly from the equations, we can derive the conserved quantities

$$\begin{aligned}
&\sigma_2 \int_{-\infty}^{\infty} Q_1(x,t)Q_1^*(-x,-t)dx + \sigma_3 \int_{-\infty}^{\infty} Q_2(x,t)Q_2^*(-x,-t)dx = \text{constant},\\
&\sigma_2 \int_{-\infty}^{\infty} Q_3(x,t)Q_3^*(-x,-t)dx + \sigma_1 \int_{-\infty}^{\infty} Q_2(x,t)Q_2^*(-x,-t)dx = \text{constant}, \qquad (116)\\
&\sigma_1 \int_{-\infty}^{\infty} Q_1(x,t)Q_1^*(-x,-t)dx - \sigma_3 \int_{-\infty}^{\infty} Q_3(x,t)Q_3^*(-x,-t)dx = \text{constant}.
\end{aligned}$$

Thus, there appears to be no positive definite conserved quantities in the above equations; in the general case there likely will be blowup solutions.

7.5. The reverse space-time multiwave reduction: $\mathsf{N}_{\ell j}(x,t) = \sigma_{\ell j}\mathsf{N}_{j\ell}(-x,-t)$

If we substitute in Eq. (106)

$$\mathsf{N}_{\ell j}(x,t) = \sigma_{\ell j}\mathsf{N}_{j\ell}(-x,-t), \qquad (117)$$

and let $x' = -x, t' = -t$ then we find

$$-(N_{j\ell,t'} - a_{lj}N_{j\ell,x'})(x',t') = \sum_{m=1}^{n}(a_{\ell m} - a_{mj})\frac{\sigma_{\ell m}\sigma_{mj}}{\sigma_{\ell j}}N_{m\ell}(x',t')N_{jm}(x',t'). \qquad (118)$$

Under the condition

$$\frac{\sigma_{\ell m}\sigma_{mj}}{\sigma_{\ell j}} = 1,$$

Eq. (118) agrees with Eq. (106) by interchanging the indices and without taking the complex conjugate.

7.6. Reverse space-time three wave equations

Under the symmetry reduction $\mathsf{N}_{21} = \sigma_1\mathsf{N}_{12}(-x,-t)$, $\mathsf{N}_{31} = \sigma_2\mathsf{N}_{13}(-x,-t)$ and $\mathsf{N}_{32} = \sigma_3\mathsf{N}_{23}(-x,-t)$, where σ_1, σ_2, and σ_3 are chosen as real numbers, we have

$$\frac{\sigma_1\sigma_3}{\sigma_2} = 1, \sigma_j = \pm 1, j = 1, 2, 3. \qquad (119)$$

As above, Eq. (106) may be put into a standard set of nonlocal three-wave interaction equations by a suitable scaling of variables. For example, we find the system

$$\begin{aligned}
&Q_{1,t}(x,t) + c_1 Q_{1,x}(x,t) = \sigma_3 Q_2(-x,-t)Q_3(-x,-t),\\
&Q_{2,t}(x,t) + c_2 Q_{2,x}(x,t) = -\sigma_2 Q_1(-x,-t)Q_3(-x,-t), \qquad (120)\\
&Q_{3,t}(x,t) + c_3 Q_{3,x}(x,t) = \sigma_1 Q_1(-x,-t)Q_2(-x,-t),
\end{aligned}$$

if we take

$$\begin{aligned}
\mathsf{N}_{12}(x,t) &= -\frac{Q_3(x,t)}{\sqrt{\beta_{13}\beta_{23}}},\\
\mathsf{N}_{31}(x,t) &= -\frac{Q_2(x,t)}{\sqrt{\beta_{12}\beta_{23}}},\\
\mathsf{N}_{23}(x,t) &= -\frac{Q_1(x,t)}{\sqrt{\beta_{12}\beta_{13}}},\\
\mathsf{N}_{13} &= \sigma_2 N_{31}(-x,-t),\\
\mathsf{N}_{32} &= \sigma_3 N_{23}(-x,-t),\\
\mathsf{N}_{21} &= \sigma_1 N_{12}(-x,-t),
\end{aligned}$$

where

$$\begin{aligned}
&\beta_{lj} := d_l - d_j = -c_l + c_j \Rightarrow d_1 = -c_1, \quad d_2 = -c_2, \quad d_3 = -c_3\\
&q_1 = -ic_2c_3,\ q_2 = -ic_1c_3,\ q_3 = -ic_1c_2,\\
&a_{12} = -c_3,\ a_{13} = -c_2,\ a_{23} = -c_1, c_3 > c_2 > c_1.
\end{aligned}$$

Directly from the equations, we can derive the conserved quantities

$$\begin{aligned}
&\sigma_2 \int_{-\infty}^{\infty} Q_1(x,t)Q_1(-x,-t)dx + \sigma_3 \int_{-\infty}^{\infty} Q_2(x,t)Q_2(-x,-t)dx = \text{constant},\\
&\sigma_2 \int_{-\infty}^{\infty} Q_3(x,t)Q_3(-x,-t)dx + \sigma_1 \int_{-\infty}^{\infty} Q_2(x,t)Q_2(-x,-t)dx = \text{constant}, \quad (121)\\
&\sigma_1 \int_{-\infty}^{\infty} Q_1(x,t)Q_1(-x,-t)dx - \sigma_3 \int_{-\infty}^{\infty} Q_3(x,t)Q_3(-x,-t)dx = \text{constant}.
\end{aligned}$$

From the above there appears to be no positive definite conserved quantities; it is expected that this set of equations will have blowup solutions. In future work, we aim to study the integrability properties of this nonlocal three-wave equation and construct soliton solutions.

8. (2+1)-dimensional space-time nonlocal multiwave and three-wave equations

In this section, we extend the analysis presented in Section 7 to two space dimensions and derive the classical (local) multiwave interaction equations and the space-time (as well as the time only) nonlocal multiwave equations. The idea is to generalize the matrix scattering problem (103) by replacing the eigenvalue term by a derivative in the transverse y coordinate. Thus, we start from the multidimensional generalized scattering problem

$$\mathbf{v}_x = \mathbf{B}\mathbf{v}_y + \mathbf{N}\mathbf{v}, \tag{122}$$

$$\mathbf{v}_t = \mathbf{C}\mathbf{v}_y + \mathbf{Q}\mathbf{v}, \tag{123}$$

where $\mathbf{v}$ is a column vector of length $n, \mathbf{B}, \mathbf{N}, \mathbf{C}$ and $\mathbf{Q}$ are $n \times n$ matrices with $\mathbf{B}$ being a real constant diagonal matrix given by $\mathbf{B} = \mathrm{d}iag(b_1, b_2, \cdots, b_n)$ and $\mathbf{N}$ is such that $\mathbf{N}_{jj} = 0, j = 1, 2, \cdots, n$. From the compatibility condition $\mathbf{v}_{xt} = \mathbf{v}_{tx}$ one finds expressions for the mixed derivatives $\mathbf{v}_{yt}$ and $\mathbf{v}_{yx}$. After setting the coefficients of the independent terms $\mathbf{v}_{yy}, \mathbf{v}_y$ and $\mathbf{v}$ to zero one finds

$$[\mathbf{C}, \mathbf{B}] = 0, \tag{124}$$

$$[\mathbf{Q}, \mathbf{B}] + [\mathbf{C}, \mathbf{N}] + \mathbf{C}_x - \mathbf{B}\mathbf{C}_y = 0, \tag{125}$$

$$\mathbf{N}_t = [\mathbf{Q}, \mathbf{N}] + \mathbf{Q}_x - \mathbf{B}\mathbf{Q}_y + \mathbf{C}\mathbf{N}_y. \tag{126}$$

With the choice

$$\mathbf{B}_{lj} = b_l \delta_{lj}, \quad \mathbf{C}_{lj} = c_l \delta_{lj}, \tag{127}$$

where b_l and c_l are taken to be real constants then Eq. (124) is satisfied. In this case, (125) yields $\mathbf{Q}_{lj} = \alpha_{lj}\mathbf{N}_{lj} \quad (l \neq l)$, where $\alpha_{lj} = \frac{c_l - c_j}{b_l - b_j} = \alpha_{jl}$. Moreover, $\mathbf{Q}_{ll} = q_l$, $q_l - q_j = ik(d_l - d_j)\alpha_{lj}$ and $\beta_{lj} = c_l - \alpha_{lj} b_l = (c_l b_j - c_j b_l)/(b_j - b_l) = \beta_{jl}$. Hence, we have the compatible two-dimensional nonlinear wave equation

$$\mathbf{N}_{lj,t} - \alpha_{lj}\mathbf{N}_{lj,x} - \beta_{lj}\mathbf{N}_{lj,y} = \sum_{m=1}^{n} (\alpha_{lm} - \alpha_{mj})\mathbf{N}_{lm}\mathbf{N}_{mj}. \tag{128}$$

8.1. Classical multiwave reduction: $\mathbf{N}_{lj}(\mathbf{x}, t) = \sigma_{lj}\mathbf{N}^*_{jl}(\mathbf{x}, t)$

For the ease of presentation we use the notation $\mathbf{x} \equiv (x, y)$. Under the classical symmetry reduction

$$\mathbf{N}_{lj}(\mathbf{x}, t) = \sigma_{lj}\mathbf{N}^*_{jl}(\mathbf{x}, t), \tag{129}$$

Ablowitz and Haberman showed that the $(2 + 1)$-dimensional system of equations (128) are compatible with its complex conjugate (recall that the

α and β coefficients are all real) so long the "σ" coefficients satisfy the constraint

$$\frac{\sigma_{lm}\sigma_{mj}}{\sigma_{lj}} = -1.$$

Thus, the symmetry condition (129) reduces the number of independent equations from $n(n-1)$ to $n(n-1)/2$. Next we show that system (128) admits novel nonlocal reductions that were not reported so far in the literature. They are of the reverse space-time nonlocal type. In the next two sections, we outline their derivations and give some conservation laws.

8.2. (2+1)-dimensional complex reverse space-time multiwave reduction: $\mathbf{N}_{lj}(\mathbf{x}, t) = \sigma_{lj}\mathbf{N}^*_{jl}(-\mathbf{x}, -t)$

If one substitutes the symmetry condition

$$\mathbf{N}_{lj}(\mathbf{x}, t) = \sigma_{lj}\mathbf{N}^*_{jl}(-\mathbf{x}, -t), \tag{130}$$

in Eq. (128) then with the help of change of variables $\mathbf{x}' = -\mathbf{x}, t' = -t$ one can show, after interchange of indices, that the system (128) is consistent with its complex conjugate (because all α_{lj}, β_{lj} are real) provided

$$\frac{\sigma_{lm}\sigma_{mj}}{\sigma_{lj}} = +1.$$

The new symmetry reduction (130) is new and, as we next see, leads to a new set of $(2+1)$-dimensional interacting nonlinear waves. For simplicity, we derive the simple and physically important case of three interacting waves.

8.3. (2+1)-dimensional complex reverse space-time three-wave equations

Here, we derive the dynamical equations governing the evolution of an interacting $(2+1)$-dimensional space-time nonlocal nonlinear waves. To do so, we explicitly write down the symmetry reduction for the case $n = 3$. They are given by

$$\mathbf{N}_{21}(\mathbf{x}, t) = \sigma_1\mathbf{N}^*_{12}(-\mathbf{x}, -t), \tag{131}$$

$$\mathbf{N}_{31}(\mathbf{x}, t) = \sigma_2\mathbf{N}^*_{13}(-\mathbf{x}, -t), \tag{132}$$

$$\mathbf{N}_{32}(\mathbf{x}, t) = \sigma_3\mathbf{N}^*_{23}(-\mathbf{x}, -t), \tag{133}$$

where, as before, all the α_{lj} and β_{lj} for $l, j = 1, 2, \cdots, n$ are real, and σ_j, $j = 1, 2, 3$ are chosen as real numbers satisfying the relation

$$\frac{\sigma_1\sigma_3}{\sigma_2} = 1, \quad \sigma_j^2 = 1 \ (j = 1, 2, 3). \tag{134}$$

Equation (128) may be put into a standard set of space-time nonlocal nonlinear interacting three-wave system by a suitable scaling of variables. With the definition

$$\mathbf{N}_{12}(\mathbf{x},t) = -\frac{Q_3(\mathbf{x},t)}{\sqrt{(-D_1+D_3)(-D_2+D_3)}}, \tag{135}$$

$$\mathbf{N}_{31}(\mathbf{x},t) = -\frac{Q_2(\mathbf{x},t)}{\sqrt{(-D_1+D_2)(-D_2+D_3)}}, \tag{136}$$

$$\mathbf{N}_{23}(\mathbf{x},t) = -\frac{Q_1(\mathbf{x},t)}{\sqrt{(-D_1+D_2)(-D_1+D_3)}}, \tag{137}$$

where

$$D_3 > D_2 > D_1 > 0,\ c_1 = -D_2D_3,\ c_2 = -D_1D_3,\ c_3 = -D_1D_2,$$
$$b_1 = -D_1,\ b_2 = -D_2,\ b_3 = -D_3,\ \alpha_{12} = -D_3,\ \alpha_{13} = -D_2,\ \alpha_{23} = -D_1,$$
$$\beta_{12} = -D_3(D_1+D_2),\ \beta_{13} = -D_2(D_1+D_3),\ \beta_{23} = -D_1(D_2+D_3).$$

we obtain the following system of three reverse space-time nonlocal interacting waves:

$$\begin{aligned} Q_{1,t}(\mathbf{x},t) + \mathbf{C}_1\cdot\nabla Q_1(\mathbf{x},t) &= \sigma_3 Q_2^*(-\mathbf{x},-t)Q_3^*(-\mathbf{x},-t),\\ Q_{2,t}(\mathbf{x},t) + \mathbf{C}_2\cdot\nabla Q_2(\mathbf{x},t) &= -\sigma_2 Q_1^*(-\mathbf{x},-t)Q_3^*(-\mathbf{x},-t),\\ Q_{3,t}(\mathbf{x},t) + \mathbf{C}_3\cdot\nabla Q_3(\mathbf{x},t) &= \sigma_1 Q_1^*(-\mathbf{x},-t)Q_2^*(-\mathbf{x},-t), \end{aligned} \tag{138}$$

where ∇ is the two dimensional gradient, $\mathbf{C}_j = (C_j^{(x)}, C_j^{(y)})$, $j = 1, 2, 3$ and

$$C_1^{(x)} = D_1,\ C_1^{(y)} = D_1(D_2+D_3),\ C_2^{(x)} = D_2,\ C_2^{(y)} = D_2(D_1+D_3),$$
$$C_3^{(x)} = D_3,\ C_3^{(y)} = D_3(D_1+D_2).$$

From the above set of dynamical equations, one can derive the following conserved quantities:

$$\begin{aligned} &\sigma_2 \int\int_{\mathbb{R}^2} Q_1(\mathbf{x},t)Q_1^*(-\mathbf{x},-t)dxdy\\ &+ \sigma_3 \int\int_{\mathbb{R}^2} Q_2(\mathbf{x},t)Q_2^*(-\mathbf{x},-t)dxdy = \text{constant}, \end{aligned} \tag{139}$$

$$\begin{aligned} &\sigma_2 \int\int_{\mathbb{R}^2} Q_3(\mathbf{x},t)Q_3^*(-\mathbf{x},-t)dxdy\\ &+ \sigma_1 \int\int_{\mathbb{R}^2} Q_2(\mathbf{x},t)Q_2^*(-\mathbf{x},-t)dxdy = \text{constant}, \end{aligned} \tag{140}$$

$$\sigma_1 \int\int_{\mathbb{R}^2} Q_1(\mathbf{x},t)Q_1^*(-\mathbf{x},-t)dxdy$$
$$- \sigma_3 \int\int_{\mathbb{R}^2} Q_3(\mathbf{x},t)Q_3^*(-\mathbf{x},-t)dxdy = \text{constant}. \tag{141}$$

Because none of the above conserved quantities is guaranteed to be positive definite, it is likely that in the general case the solution will blowup in finite time. This would be an interesting future direction to consider.

8.4. (2+1)-dimensional reverse space-time multiwave reduction: $\mathbf{N}_{lj}(\mathbf{x},t) = \sigma_{lj}\mathbf{N}_{jl}(-\mathbf{x},-t)$

Another interesting symmetry reduction that Eq. (128) admits is given by

$$\mathbf{N}_{lj}(\mathbf{x},t) = \sigma_{lj}\mathbf{N}_{jl}(-\mathbf{x},-t), \tag{142}$$

which would result in a reduction of the number of equations from $n(n-1)$ to $n(n-1)/2$. Indeed, substituting (142) into (128); make the change of variables $\mathbf{x}' = -\mathbf{x}, t' = -t$ and upon rearrangement of indices, one obtain the same Eq. (128) provided

$$\frac{\sigma_{lm}\sigma_{mj}}{\sigma_{lj}} = 1. \tag{143}$$

With the help of the symmetry condition (142) we will next derive the reverse space-time nonlocal interacting three-wave system following the same idea we outlined in Section 8.3.

8.5. (2+1)-dimensional reverse space-time three-wave equations

Under the symmetry reduction $\mathbf{N}_{21}(\mathbf{x},t) = \sigma_1\mathbf{N}_{12}(-\mathbf{x},-t)$, $\mathbf{N}_{31}(\mathbf{x},t) = \sigma_2\mathbf{N}_{13}(-\mathbf{x},-t)$ and $\mathbf{N}_{32}(\mathbf{x},t) = \sigma_3\mathbf{N}_{23}(-\mathbf{x},-t)$, where σ_1, σ_2, and σ_3 are chosen as real numbers, we have $\sigma_1\sigma_3/\sigma_2 = 1$ with $\sigma_j^2 = 1$ $(j = 1,2,3)$. Equation (128) may be put into a standard set of nonlocal three-wave interaction equations by a suitable scaling of variables. For example, we find the system

$$\begin{aligned}
&Q_{1,t}(\mathbf{x},t) + \mathbf{C}_1 \cdot \nabla Q_1(\mathbf{x},t) = \sigma_3 Q_2(-\mathbf{x},-t)Q_3(-\mathbf{x},-t),\\
&Q_{2,t}(\mathbf{x},t) + \mathbf{C}_2 \cdot \nabla Q_2(\mathbf{x},t) = -\sigma_2 Q_1(-\mathbf{x},-t)Q_3(-\mathbf{x},-t),\\
&Q_{3,t}(\mathbf{x},t) + \mathbf{C}_3 \cdot \nabla Q_3(\mathbf{x},t) = \sigma_1 Q_1(-\mathbf{x},-t)Q_2(-\mathbf{x},-t),
\end{aligned} \tag{144}$$

if we define the following new functions

$$\mathbf{N}_{12}(\mathbf{x},t) = -\frac{Q_3(\mathbf{x},t)}{\sqrt{(-D_1+D_3)(-D_2+D_3)}}, \tag{145}$$

$$\mathbf{N}_{31}(\mathbf{x},t) = -\frac{Q_2(\mathbf{x},t)}{\sqrt{(-D_1+D_2)(-D_2+D_3)}}, \tag{146}$$

$$\mathbf{N}_{23} = -\frac{Q_1}{\sqrt{(-D_1+D_2)(-D_1+D_3)}}, \tag{147}$$

$$\mathbf{N}_{13}(\mathbf{x},t) = \sigma_2 N_{31}^*(-\mathbf{x},-t), \tag{148}$$

$$\mathbf{N}_{32}\mathbf{x},t) = \sigma_3 N_{23}^* {-\mathbf{x}},-t), \tag{149}$$

$$\mathbf{N}_{21}\mathbf{x},t) = \sigma_1 N_{12}^* {-\mathbf{x}},-t), \tag{150}$$

where we have defined $\mathbf{C}_j \equiv (C_j^{(x)}, C_j^{(y)})$, $j = 1, 2, 3$ and

$$C_1^{(x)} = D_1,\ C_1^{(y)} = D_1(D_2+D_3),\ C_2^{(x)} = D_2,\ C_2^{(y)} = D_2(D_1+D_3),$$
$$C_3^{(x)} = D_3,\ C_3^{(y)} = D_3(D_1+D_2),$$

$$D_3 > D_2 > D_1 > 0,\ c_1 = -D_2D_3,\ c_2 = -D_1D_3,\ c_3 = -D_1D_2,$$
$$b_1 = -D_1,\ b_2 = -D_2,\ b_3 = -D_3,\ \alpha_{12} = -D_3,\ \alpha_{13} = -D_2,\ \alpha_{23} = -D_1,$$
$$\beta_{12} = -D_3(D_1+D_2),\ \beta_{13} = -D_2(D_1+D_3),\ \beta_{23} = -D_1(D_2+D_3).$$

As was done before, we can derive the following conserved quantities:

$$\sigma_2 \int\int_{\mathbb{R}^2} Q_1(\mathbf{x},t)Q_1(-\mathbf{x},-t)dxdy + \sigma_3 \int\int_{\mathbb{R}^2} Q_2(\mathbf{x},t)Q_2(-\mathbf{x},-t)dxdy = \text{constant}, \tag{151}$$

$$\sigma_2 \int\int_{\mathbb{R}^2} Q_3(\mathbf{x},t)Q_3(-\mathbf{x},-t)dxdy + \sigma_1 \int\int_{\mathbb{R}^2} Q_2(\mathbf{x},t)Q_2(-\mathbf{x},-t)dxdy = \text{constant}, \tag{152}$$

$$\sigma_1 \int\int_{\mathbb{R}^2} Q_1(\mathbf{x},t)Q_1(-\mathbf{x},-t)dxdy - \sigma_3 \int\int_{\mathbb{R}^2} Q_3(\mathbf{x},t)Q_3(-\mathbf{x},-t)dxdy = \text{constant}. \tag{153}$$

As with the complex reverse space-time nonlocal three-wave system, none of the above conserved quantity appears to be positive definite. It would be interesting to see if the above three wave system can develop a finite time singularity.

9. Integrable nonlocal discrete NLS models: Reverse discrete-time, reverse time, and PT preserved symmetries

In this section, we derive discrete analogues to the nonlocal NLS equations (3) and (3). The resulting models are integrable and admit infinite number of conserved quantities. Our approach is based on the integrable discrete scattering problem [37]

$$v_{n+1} = \begin{pmatrix} z & Q_n \\ R_n & z^{-1} \end{pmatrix} v_n, \tag{154}$$

$$\frac{dv_n}{dt} = \begin{pmatrix} \mathsf{A}_n & \mathsf{B}_n \\ \mathsf{C}_n & \mathsf{D}_n \end{pmatrix} v_n, \tag{155}$$

where $v_n = (v_n^{(1)}, v_n^{(2)})^T$, Q_n and R_n vanish rapidly as $n \to \pm\infty$ and z is a complex spectral parameter. Here,

$$\mathsf{A}_n = iQ_nR_{n-1} - \frac{i}{2}\left(z - z^{-1}\right)^2, \tag{156}$$

$$\mathsf{B}_n = -i\left(zQ_n - z^{-1}Q_{n-1}\right), \tag{157}$$

$$\mathsf{C}_n = i\left(z^{-1}R_n - zR_{n-1}\right) \tag{158}$$

$$\mathsf{D}_n = -iR_nQ_{n-1} + \frac{i}{2}\left(z - z^{-1}\right)^2. \tag{159}$$

The discrete compatibility condition $\frac{d}{dt}v_{n+1} = (\frac{d}{dt}v_m)_{m=n+1}$ yields

$$i\frac{d}{dt}Q_n(t) = \Delta_n Q_n(t) - Q_n(t)R_n(t)\left[Q_{n+1}(t) + Q_{n-1}(t)\right], \tag{160}$$

$$-i\frac{d}{dt}R_n(t) = \Delta_n R_n(t) - Q_n(t)R_n(t)\left[R_{n+1}(t) + R_{n-1}(t)\right], \tag{161}$$

where

$$\Delta_n F_n \equiv F_{n+1} - 2F_n + F_{n-1}. \tag{162}$$

In [37], it was shown that the system of equations (160) and (161) are compatible under the symmetry reduction

$$R_n(t) = \sigma Q_n^*(t), \quad \sigma = \mp 1, \tag{163}$$

and gives rise to the Ablowitz–Ladik model [37,38]

$$i\frac{dQ_n(t)}{dt} = \Delta_n Q_n - \sigma |Q_n(t)|^2 \left[Q_{n+1}(t) + Q_{n-1}(t)\right]. \tag{164}$$

9.1. Reverse discrete-time reduction: $R_n(t) = \sigma Q_{-n}(-t)$

Interestingly, the system of discrete equations (160) and (161) are compatible under the symmetry reduction

$$R_n(t) = \sigma Q_{-n}(-t), \quad \sigma = \mp 1, \tag{165}$$

and gives rise to the reverse discrete-time nonlocal discrete NLS equation:

$$i\frac{dQ_n(t)}{dt} = \Delta_n Q_n - \sigma Q_n(t) Q_{-n}(-t) \left[Q_{n+1}(t) + Q_{n-1}(t)\right]. \tag{166}$$

The discrete symmetry constraint (165) is new and was not noticed in the literature. Because Eq. (166) comes out of the Ablowitz–Ladik scattering problem, as such, it constitutes an infinite dimensional integrable Hamiltonian dynamical system. The first two conserved quantities are given by

$$\sum_{n=-\infty}^{+\infty} Q_n(t) Q_{1-n}(-t) = \text{constant}. \tag{167}$$

$$\sum_{n=-\infty}^{+\infty} \left[\sigma Q_n(t) Q_{2-n}(-t) - \frac{1}{2}\left(Q_n(t) Q_{1-n}(-t)\right)^2\right] = \text{constant}. \tag{168}$$

$$\prod_{n=-\infty}^{+\infty} \left[1 - \sigma Q_n(t) Q_{-n}(-t)\right] = \text{constant}. \tag{169}$$

Importantly, Eq. (166) is a Hamiltonian dynamical system with $Q_n(t)$ and $Q_{-n}(-t)$ playing the role of coordinates and conjugate momenta, respectively. The corresponding Hamiltonian and (the noncanonical) brackets are given by

$$H = -\sigma \sum_{n=-\infty}^{+\infty} Q_{-n}(-t)\left(Q_{n+1}(t) + Q_{n-1}(t)\right) \tag{170}$$

$$-2\sum_{n=-\infty}^{+\infty} \log\left(1-\sigma Q_n(t)Q_{-n}(-t)\right).$$

$$\{Q_m(t), Q_{-n}(-t)\} = i\sigma\left(1-\sigma Q_n(t)Q_{-n}(-t)\right)\delta_{n,m}. \qquad (171)$$

$$\{Q_n(t), Q_m(t)\} = \{Q_n(t), Q_{-m}(-t)\} = 0. \qquad (172)$$

9.2. Reverse time discrete symmetry: $R_n(t) = \sigma Q_n(-t)$

Equations (160) and (161) admit another important symmetry reduction given by

$$R_n(t) = \sigma Q_n(-t), \quad \sigma = \mp 1. \qquad (173)$$

This symmetry reduction is called reverse time Ablowitz–Ladik symmetry and results in the following discrete reverse time nonlocal discrete NLS equation:

$$i\frac{dQ_n(t)}{dt} = \Delta_n Q_n - \sigma Q_n(t)Q_n(-t)\left[Q_{n+1}(t) + Q_{n-1}(t)\right]. \qquad (174)$$

The discrete symmetry constraint (173) is also new and was not noticed in the literature so far. As is the case with the complex discrete-time symmetry, Eq. (174) is also integrable and posses an infinite number of conservation laws. The first few conserved quantities are listed below

$$\sum_{n=-\infty}^{+\infty} Q_n(t)Q_{n-1}(-t) = \text{constant}. \qquad (175)$$

$$\sum_{n=-\infty}^{+\infty}\left[\sigma Q_n(t)Q_{n-2}(-t) - \frac{1}{2}\left(Q_n(t)Q_{n-1}(-t)\right)^2\right] = \text{constant}. \qquad (176)$$

$$\prod_{n=-\infty}^{+\infty}\left[1-\sigma Q_n(t)Q_n(-t)\right] = \text{constant}. \qquad (177)$$

Importantly, Eq. (174) is a Hamiltonian dynamical system with $Q_n(t)$ and $Q_{-n}(-t)$ playing the role of coordinates and conjugate momenta, respectively. The corresponding Hamiltonian and (the noncanonical) brackets are given by

$$H = -\sigma\sum_{n=-\infty}^{+\infty} Q_n(-t)\left(Q_{n+1}(t) + Q_{n-1}(t)\right) \qquad (178)$$

$$-2\sum_{n=-\infty}^{+\infty}\log\left(1-\sigma Q_n(t)Q_n(-t)\right).$$

$$\{Q_m(t), Q_n(-t)\} = i\sigma\left(1-\sigma Q_n(t)Q_{-n}(-t)\right)\delta_{n,m}. \tag{179}$$

$$\{Q_n(t), Q_m(t)\} = \{Q_n(t), Q_m(-t)\} = 0. \tag{180}$$

In summary, the discrete systems (160) and (160) admit four different symmetry reduction:

1. Standard Ablowitz–Ladik symmetry
$$R_n(t) = \sigma Q_n^*(t), \quad \sigma = \mp 1, \tag{181}$$
giving rise to the so-called Ablowitz–Ladik model (164).
2. Reverse discrete-time symmetry
$$R_n(t) = \sigma Q_{-n}(-t), \quad \sigma = \mp 1, \tag{182}$$
giving rise to Eq. (166),
3. Reverse time discrete symmetry
$$R_n(t) = \sigma Q_n(-t), \quad \sigma = \mp 1, \tag{183}$$
giving rise to Eq. (174),
4. Discrete PT preserved symmetry
$$R_n(t) = \sigma Q_{-n}^*(t), \sigma = \mp 1, \tag{184}$$
giving rise to the discrete PT symmetric integrable nonlocal discrete NLS equation first found in [20]:
$$i\frac{dQ_n(t)}{dt} = \Delta_n Q_n - \sigma Q_n(t)Q_{-n}^*(t)\left[Q_{n+1}(t)+Q_{n-1}(t)\right]. \tag{185}$$

10. IST: 2 × 2 AKNS type

Many of the above reverse space-time nonlocal evolution equations introduced in this paper came out of crucial symmetry reductions of general AKNS scattering problem (21)–(24). As such, they constitute infinite-dimensional integrable Hamiltonian dynamical systems which are solvable by the IST. The method of solution involves three major steps: (i) direct scattering problem which involves finding the associated eigenfunctions, scattering data and their symmetries, (ii) identifying the time evolution of the scattering data, and (iii) solving the inverse problem using the Riemann–Hilbert approach or other inverse methods. In what follows we highlight the main results behind each step for the AKNS scattering problem given in (22) subject to the new reversed space-time symmetry reductions. The full account of the inverse scattering theory for all evolution equations introduced in this paper is beyond the scope of this paper and will be discussed in future work.

10.1. Direct scattering problem

The analysis presented in this paper assumes that the potential functions $q(x,t)$ and $r(x,t)$ decay to zero sufficiently fast at infinity. Thus, solutions of the scattering problem (21) are defined and satisfy the boundary conditions

$$\begin{aligned} \phi &\sim \begin{pmatrix}1\\0\end{pmatrix} e^{-ikx}, \quad \overline{\phi} \sim \begin{pmatrix}0\\1\end{pmatrix} e^{ikx}, \text{ as } x \to -\infty \\ \psi &\sim \begin{pmatrix}0\\1\end{pmatrix} e^{ikx}, \quad \overline{\psi} \sim \begin{pmatrix}1\\0\end{pmatrix} e^{-ikx}, \text{ as } x \to +\infty. \end{aligned} \tag{186}$$

Note that bar does not denote complex conjugation; we use $*$ to denote complex conjugation. It is expedient to define new functions

$$M(x,t,k) = e^{ikx}\phi(x,t,k), \ \overline{M}(x,t,k) = e^{-ikx}\overline{\phi}(x,t,k), \tag{187}$$

$$N(x,t,k) = e^{-ikx}\psi(x,t,k), \ \overline{N}(x,t,k) = e^{ikx}\overline{\psi}(x,t,k), \tag{188}$$

with

$$\begin{aligned} M &\sim \begin{pmatrix}1\\0\end{pmatrix}, \quad \overline{M} \sim \begin{pmatrix}0\\1\end{pmatrix}, \text{ as } x \to -\infty \\ N &\sim \begin{pmatrix}0\\1\end{pmatrix}, \quad \overline{N} \sim \begin{pmatrix}1\\0\end{pmatrix}, \text{ as } x \to +\infty \end{aligned} \tag{189}$$

that satisfy constant boundary conditions at infinity and reformulate the direct scattering problem in terms of them. With this at hand, when the potentials q, r are integrable (i.e., they are in class L^1) one can derive an integral equation for the above functions and use them to show that $M(x,t,k), N(x,t,k)$ are analytic functions in the upper half complex k plane whereas $\overline{M}(x,t,k), \overline{N}(x,t,k)$ are analytic functions in the lower half complex k plane [29]. The solutions $\phi(x,t,k)$ and $\overline{\phi}(x,t,k)$ of the scattering problem (21) with the boundary conditions (186) are linearly independent. The same hold for $\psi(x,t,k)$ and $\overline{\psi}(x,t,k)$. We denote by

$\Phi(x,t,k) \equiv (\phi(x,t,k),\ \overline{\phi}(x,t,k))$ and $\Psi(x,t,k) \equiv (\overline{\psi}(x,t,k),\ \psi(x,t,k))$. Clearly, these two set of functions are linearly dependent and write

$$\Phi(x,t,k) = S(k,t)\Psi(x,t,k), \tag{190}$$

where $S(k,t)$ is the scattering matrix given by

$$S(k,t) = \begin{pmatrix} a(k,t) & b(k,t) \\ \overline{b}(k,t) & \overline{a}(k,t) \end{pmatrix}. \tag{191}$$

The elements of the scattering matrix $S(k,t)$ are related to the Wronskian of the system via the relations

$$a(k,t) = W(\phi(x,t,k), \psi(x,t,k)), \tag{192}$$

$$\overline{a}(k,t) = W(\overline{\psi}(x,t,k), \overline{\phi}(x,t,k)), \tag{193}$$

and

$$b(k,t) = W(\overline{\psi}(x,t,k), \phi(x,t,k)), \tag{194}$$

$$\overline{b}(k,t) = W(\overline{\phi}(x,t,k), \psi(x,t,k)), \tag{195}$$

where $W(u,v)$ is the Wronskian of the two solutions u, v and is given by $W(u,v) = u_1 v_2 - v_1 u_2$ where in terms of components $u = [u_1, u_2]^T$ where T represents the transpose. Moreover, it can be shown that $a(k), \overline{a}(k)$ are respectively analytic functions in the upper/lower half complex k plane. However $b(k)$ and $\overline{b}(k)$ are generally not analytic anywhere.

10.2. Inverse scattering problem

The inverse problem consists of constructing the potential functions $r(x,t)$ and $q(x,t)$ from the scattering data (reflection coefficients), e.g., $\rho(k,t) = e^{-4ik^2t} b(k,0)/a(k,0)$ and $\overline{\rho}(k,t) = e^{4ik^2t}\overline{b}(k,0)/\overline{a}(k,0)$ defined on $\mathrm{Im}k = 0$ as well as the eigenvalues $k_j, \overline{k}_j$ and norming constants (in x) $C_j(t), \overline{C}_j(t)$. Using the Riemann–Hilbert approach, from Eq. (190) one can find equations governing the eigenfunctions $N(x,t,k), \overline{N}(x,t,k)$ [29]

$$\overline{N}(x,t,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j(t) e^{2ik_j x} N(x,t,k_j)}{k - k_j} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\zeta,t) e^{2i\zeta x} N(x,t,\zeta)}{\zeta - (k - i0)} d\zeta, \tag{196}$$

$$N(x,t,k) = \begin{pmatrix}0\\1\end{pmatrix} + \sum_{j=1}^{\overline{J}} \frac{\overline{C}_j(t)e^{-2i\overline{k}_j x}\overline{N}(x,t,\overline{k}_j)}{k-\overline{k}_j}$$
$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}(t)(\zeta)e^{-2i\zeta x}\overline{N}(x,t,\zeta)}{\zeta-(k+i0)}d\zeta. \tag{197}$$

To close the system we substitute $k = \overline{k}_\ell$ and $k = k_\ell$ in (197) and (196), respectively, and obtain a linear algebraic integral system of equations that solve the inverse problem for the eigenfunctions $N(x,t,k)$ and $\overline{N}(x,t,k)$. In the case with zero reflection coefficient, i.e., $\rho(t) = \overline{\rho}(t) = 0$ the resulting algebraic system governing the soliton solution reads

$$\overline{N}(x,t,\overline{k}_\ell) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{J} \frac{C_j(t)e^{2ik_j x}N(x,t,k_j)}{\overline{k}_\ell - k_j}, \tag{198}$$

$$N(x,t,k_\ell) = \begin{pmatrix}0\\1\end{pmatrix} + \sum_{j=1}^{\overline{J}} \frac{\overline{C}_j(t)e^{-2i\overline{k}_j x}\overline{N}(x,t,\overline{k}_j)}{k_\ell - \overline{k}_j}. \tag{199}$$

10.3. Recovery of the potentials

To reconstruct the potentials for all time: $q(x,t)$, $r(x,t)$ we compare the asymptotic expansions of Eq. (197) and (196) to the Jost functions and find (for pure soliton solution only)

$$q(x,t) = 2i\sum_{\ell=1}^{\bar{J}} \bar{C}_\ell(t)e^{-2i\bar{k}_\ell x}\bar{N}_1(x,\bar{k}_\ell). \tag{200}$$

$$r(x,t) = -2i\sum_{\ell=1}^{J} C_\ell(t)e^{2ik_\ell x}N_2(x,k_\ell). \tag{201}$$

Once all the symmetries of the scattering data are known, we can obtain the solution q, which satisfies the spatial symmetries by solving the above equations.

10.4. Evolution of the scattering data

The time dependence of the potentials q and r in Eq. (200) and (201) is encoded in the eigenvalues and norming constants C_j and $\overline{C}_j$. Their time evolution is derived from Eq. (23) and (24). The space, time and space-time nonlocal NLS, mKdV and SG equations belong to the same hierarchy, i.e., they all originate from the same scattering problem (21) with different A, B and C which in turn determines the time evolution of the scattering data and norming constants. For the problems we will be studying in detail here, following the derivation outlined in [29] for the temporal evolution one finds the following: In all cases we have

$$a(k,t) = a(k,0), \bar{a}(k,t) = \bar{a}(k,0),$$

so that the zero's of $a(k)$ and $\bar{a}(k)$, denoted by, $k_j, \bar{k}_j$, $j = 1, 2...J$, respectively are constant in time. For NLS and nonlocal NLS problems

$$b^{\mathrm{NLS}}(k,t) = b(k,0)e^{-4ik^2t},$$

$$\bar{b}^{\mathrm{NLS}}(k,t) = \bar{b}(k,0)e^{4ik^2t}, k \in \mathbb{R},$$

$$C_j^{\mathrm{NLS}}(t) = C_j(0)e^{-4ik_j^2t}, \tag{202}$$

$$\overline{C}_j^{\mathrm{NLS}}(t) = \overline{C}_j(0)e^{4i\overline{k}_j^2t}. \tag{203}$$

Here, k_j and $\overline{k}_j$ are often called the soliton eigenvalues and $C_j(0)$, $\bar{C}_j(0)$ are termed norming constants. For mKdV and nonlocal mKdV problems

$$b^{\mathrm{mKdV}}(k,t) = b(k,0)e^{8ik^3t},$$

$$\bar{b}^{\mathrm{mKdV}}(k,t) = \bar{b}(k,0)e^{-8ik^3t}, k \in \mathbb{R},$$

$$C_j^{\mathrm{mKdV}}(t) = C_j(0)e^{8ik_j^3t}, \tag{204}$$

$$\overline{C}_j^{\mathrm{mKdV}}(t) = \overline{C}_j(0)e^{-8i\overline{k}_j^3t}, \tag{205}$$

and for the sine-Gordon (sG) equation we have

$$C_j^{\mathrm{sG}}(t) = C_j(0)e^{-it/(2k_j)}, \tag{206}$$

$$\overline{C}_j^{\mathrm{sG}}(t) = \overline{C}_j(0)e^{it/(2\overline{k}_j)}. \tag{207}$$

In the latter equations we used the boundary condition (83).

11. Symmetries and soliton solutions

In this section, we construct soliton solutions to the time and space-time nonlocal NLS as well as the mKdV and sine-Gordon (sG) equations. Pure soliton solutions correspond to zero reflection coefficients, i.e., $\rho(\xi, t) = 0$ and $\bar{\rho}(\xi, t) = 0$ for all real ξ. In this case the system (196), (197) reduces to an algebraic equations (199) and (198) supplemented by the time dependence (202-205) that determine the functional form of the solitons for the nonlocal NLS, mKdV, and sG equations. Next, we obtain a one-soliton solution of the $N, \bar{N}$ equations (198) and (199) by taking $J = \bar{J} = 1$ to find

$$N_2(x,t) = \bar{N}_1(x,t) = \frac{1}{1 + \frac{C_1(t)\bar{C}_1(t)}{(k_1 - \bar{k}_1)^2 e^{-2i(k_1 - \bar{k}_1)x}}}. \tag{208}$$

The corresponding potentials (200)–(201) are given by

$$q(x,t) = \frac{2ie^{-2i\bar{k}_1 x}\bar{C}_1(t)}{1 + \frac{C_1(t)\bar{C}_1(t)}{(k_1 - \bar{k}_1)^2 e^{-2i(k_1 - \bar{k}_1)x}}}, \tag{209}$$

$$r(x,t) = -\frac{2ie^{2ik_1 x}C_1(t)}{1 + \frac{C_1(t)\bar{C}_1(t)}{(k_1 - \bar{k}_1)^2 e^{-2i(k_1 - \bar{k}_1)x}}}. \tag{210}$$

Below, for the 2×2 AKNS scattering problem we will give the relevant symmetries and (for simplicity) their associated one-soliton solutions considered in this paper.

11.1. Standard AKNS symmetry: $r(x,t) = \sigma q^*(x,t)$

The original symmetry (associated with solitons) considered in [12] was

$$r(x,t) = \sigma q^*(x,t), \tag{211}$$

where we recall $\sigma = \mp 1$. The (additional) time dependence of the scattering data associated with the classical NLS equation is

$$\bar{b}(k,t) = \sigma b^*(k,t),\, k \in \mathbb{R},$$

and for $\sigma = -1$

$$\bar{C}_j(t) = -C_j^*(t),\, j = 1, 2..., J.$$

The corresponding continuous and discrete symmetries in scattering space, at the initial time, are given by

$$\bar{a}(k,0) = a^*(k^*,0),\, \bar{b}(k,0) = \sigma b^*(k,0),\, k \in \mathbb{R}$$

$$\sigma = -1 : \bar{k}_j = k_j^*,\, \bar{C}_j(0) = -C_j^*(0),\, j = 1, 2..., J \tag{212}$$

The above symmetries allow us to formulate the general linearization of the classical NLS equation (1) with the reduction (211) given above. Then the corresponding well-known one soliton solution of the classical NLS equation (1) is obtained from Eqs. (209) and (210) with $J = 1, k_1 = \xi + i\eta$; it is given by

$$q_{NLS}(x,t) = 2\eta \operatorname{sech}(2\eta(x - 4\xi t - x_0))e^{-2i\xi x + 4i(\xi^2 - \eta^2)t - i\psi_0}, \tag{213}$$

where $e^{2\eta x_0} = |C_1(0)|/(2\eta)$, $\psi_0 = arg(C_1(0)) - \pi/2$. We also note that the above symmetries in scattering space imply that $r(x,t)$ given by Eq. (210) automatically satisfy the physical symmetry (211).

11.2. Reverse time AKNS symmetry: $r(x,t) = \sigma q(x,-t), q \in \mathbb{C}$

The solution corresponding to the physical symmetry

$$r(x,t) = \sigma q(x,-t), \tag{214}$$

of the corresponding nonlocal in time NLS equation (3) can be obtained by employing the following temporal symmetries in scattering space:

$$\bar{b}(k,t) = -\sigma b(-k,-t),$$

$$\bar{C}(\bar{k}_j, t) = C(k_j, -t), \sigma = -1,$$

and we denote

$$\bar{C}(\bar{k}_j, t) = \bar{C}_j(t) \,\text{and}\, C(k_j, t) = C_j(t).$$

The symmetries at $t = 0$ satisfy

$$\bar{a}(k,0) = -a^*(-k,0), \bar{b}(k,0) = -\sigma b(-k,0), k \in \mathbb{R}, \tag{215}$$

$$\sigma = -1 : \bar{k}_j = -k_j, \bar{C}_j(0) = C_j(0), j = 1, 2, ...J. \tag{216}$$

Further details of how to obtain these symmetries are given in the Appendix (see also [6]). With the symmetries: $\bar{k}_1 = -k_1$ and $\bar{C}_1(0) = C_1(0)$ and using the above time dependence for $C_1(t), \bar{C}_1(t)$ the nonlocal in time NLS equation (3) has the following one soliton solution

$$q_{TNLS}(x,t) = \frac{2iC_1(0)e^{2ik_1x}e^{4ik_1^2t}}{1 + \dfrac{C_1^2(0)}{4k_1^2}e^{4ik_1x}}, \tag{217}$$

$$r_{TNLS}(x,t) = -\frac{2iC_1(0)e^{2ik_1x}e^{-4ik_1^2t}}{1 + \dfrac{C_1^2(0)}{4k_1^2}e^{4ik_1x}}. \tag{218}$$

One can see that the symmetry condition $r(x,t) = -q(x,-t)$ is automatically satisfied. With $k_1 = \xi + i\eta$ another form of the solution is

$$q_{TNLS}(x,t) = \frac{2iC_1(0)e^{2i\xi x}e^{4i(\xi^2-\eta^2)t}e^{-2\eta x}e^{-8\xi\eta t}}{1 + \frac{C_1^2(0)}{4k_1^2}e^{4i\xi x}e^{-4\eta x}}. \tag{219}$$

Note that as $|x| \to \infty$, $q_{TNLS}(x,t) \to 0$, but as $\xi t \to -\infty$, $q_{TNLS}(x,t) \to \infty$ so in general it is an unstable solution. If we write

$$\frac{C_1(0)}{2k_1} = e^{2\eta x_0}e^{-2i\psi_0},$$

then a singularity can occur when

$$1 + e^{4i(\xi x - \psi_0)}e^{-4\eta(x-x_0)} = 0,$$

or when

$$x = x_0, 4(\xi x_0 - \psi_0) = (2n+1)\pi, n \in \mathbb{Z}.$$

When we take a special case: $\xi = 0$ the solution is stable; it can be singular depending on $C_1(0)$; but if we further take $C_1(0) = |C_1(0)|$ so that $\psi_0 = 0$, and call $|C_1(0)|/(2\eta) = e^{-2\eta x_0}$ we find

$$q_{TNLSR}(x,t) = 2\eta\text{sech}[2\eta(x-x_0)]e^{4i\eta^2 t}, \tag{220}$$

which is not singular. We note that from Eq. (213) the one soliton solution of NLS with $\xi = 0$ is given by

$$q_{TNLS}(x,t) = 2\eta\text{sech}(2\eta(x-x_0))e^{-4i\eta^2 t - i\psi_0}, \tag{221}$$

which is the same solution as given above in Eq. (220) but with $\psi_0 = 0$. Indeed, $\psi_0 = 0$ is necessary for this to be a solution of Eq. (3). Indeed any solution to the classical NLS (1) that satisfies the property

$$q^*(x,t) = q(x,-t), \tag{222}$$

automatically satisfies the corresponding nonlocal (in time) NLS equation (3). This holds when the solution (221) obeys $\psi_0 = 0$. In this regard, we also note that the solution

$$q(x,t) = \eta\tanh(\eta x)e^{2i\eta^2 t}, \tag{223}$$

with nonzero boundary conditions $q(x,t) \sim \pm\eta e^{2i\eta^2 t}$ as $x \to \pm\infty$, which is a "dark" soliton solution of the classical NLS equation (1), solves Eq. (3) with $\sigma = 1$.

11.3. *PT Symmetry:* $r(x,t) = \sigma q^*(-x,t)$

The physical PT symmetry (associated with solitons) considered in [14, 22] was

$$r(x,t) = \sigma q^*(-x,t). \tag{224}$$

The corresponding continuous and discrete symmetries in scattering space are given by

$$a(k,t) = a^*(-k^*,t) = a(k,0), \bar{a}(k,t) = \bar{a}^*(-k^*,t) = \bar{a}(k,0), \tag{225}$$

$$\bar{b}(k,t) = \sigma b^*(-k,t), k \in \mathbb{R}. \tag{226}$$

When $\sigma = -1$ there are soliton eigenvalues

$$k_j = -k_j^*, \bar{k}_j = -\bar{k}_j^*, j = 1, 2..., J.$$

We calculate the norming constants from

$$C_j(0) = b_j/a'(kj), b_j = e^{i\theta_j}, \theta_j \in \mathbb{R}, j = 1, 2..., J,$$

$$\bar{C}_j(0) = \bar{b}_j/\bar{a}'(\bar{k}_j), \bar{b}_j = e^{i\bar{\theta}_j}, \bar{\theta}_j \in \mathbb{R}, j = 1, 2..., J, \tag{227}$$

and the terms $a'(kj), \bar{a}'(\bar{k}_j)$ are computed via the trace formulae [22]. When $J = 1$ the eigenvalues are on the imaginary axis: $k_1 = i\eta, \bar{k}_1 = -i\bar{\eta}, \eta > 0, \bar{\eta} > 0$; then the trace formulae gives

$$C_1(0) = i(\eta + \bar{\eta})e^{i\theta}, \bar{C}_1(0) = -i(\eta + \bar{\eta})e^{i\bar{\theta}}, \tag{228}$$

the one-soliton solution of the PT symmetric nonlocal NLS equation (2) with the reduction

$$r(x,t) = \sigma q^*(-x,t),$$

is found to be

$$q_{PT}(x,t) = \frac{2(\eta + \bar{\eta})e^{i\bar{\theta}}e^{-2\bar{\eta}x - 4i\bar{\eta}^2 t}}{1 - e^{i(\theta + \bar{\theta})}e^{-2(\eta + \bar{\eta})x + 4i(\eta^2 - \bar{\eta}^2 t)}}. \tag{229}$$

An alternative form of writing the above one-soliton solution (229) is

$$q(x,t) = \frac{(\eta + \bar{\eta})e^{i(\bar{\theta} - \theta - \pi)/2}e^{-(\bar{\eta} - \eta)x}e^{-2i(\eta^2 + \bar{\eta}^2)t}}{\cosh\left[(\eta + \bar{\eta})x - 2i(\eta^2 - \bar{\eta}^2)t - i(\theta + \bar{\theta} + \pi)/2\right]}. \tag{230}$$

Next, some remarks are in order.

- The solution $q(x,t)$ given in (229) is doubly periodic in time with periods given by $T_1 = \frac{\pi}{2\bar{\eta}^2}$ and $T_2 = \frac{\pi}{2(\eta^2 - \bar{\eta}^2)}$.

- The intensity $|q(x,t)|^2$ breathes in time with period given by $T = \frac{\pi}{2(\eta^2-\bar{\eta}^2)}$.
- The solution (229) can develop a singularity in finite time. Indeed, at the origin ($x=0$) the solution (230) becomes singular when

$$t_n = \frac{2n\pi - (\theta + \overline{\theta})}{4(\eta^2 - \overline{\eta}^2)}, n \in \mathbb{Z}. \tag{231}$$

- The solution (229) is characterized by two important time scales: the singularity time scale and the periodicity of breathing.
- A feature of the solution (229) (and other singular solutions discussed in this paper) is that it can be defined after singularity has developed; i.e., it has a pole in time and it can be avoided in the complex time plane; i.e., the solution is of Painlevé type.
- We recall that not all members of the one-soliton family develop a singularity at finite time. Indeed, if one let $\eta = \bar{\eta} \equiv \eta$ in (229) then we arrive at the well-behaved soliton solution of the nonlocal PT symmetric NLS equation (2)

$$q(x,t) = 2\eta \text{sech}[2\eta x - i\theta] e^{-4i\eta^2 t}, \tag{232}$$

where η and θ are arbitrary real constants.

Note that when $\theta \neq 0$ the soliton given (232) is not a solution to the classical (local) NLS equation (1). The PT symmetric induced potential is given by (see Eq. (3))

$$V \equiv q(x,t)q^*(-x,t) = 4\eta^2 \text{sech}^2[2\eta x - i\theta]. \tag{233}$$

The real and imaginary parts of the induced potential are, respectively, given by

$$V_R = \frac{4\eta^2 \left[\cos^2\theta \cosh^2(2\eta x) - \sin^2\theta \sinh^2(2\eta x)\right]}{\left[\cos^2\theta \cosh^2(2\eta x) + \sin^2\theta \sinh^2(2\eta x)\right]^2},$$

$$V_I = \frac{\sin(2\theta)\sinh(4\eta x)}{2\left[\cos^2\theta \cosh^2(2\eta x) + \sin^2\theta \sinh^2(2\eta x)\right]^2}.$$

11.4. Reverse space-time symmetry: $r(x,t) = \sigma q(-x,-t), q \in \mathbb{C}$

The corresponding continuous and discrete symmetries in scattering space are given by

$$\bar{b}(k,t) = \sigma b(k,-t), k \in \mathbb{R}. \tag{234}$$

When $\sigma = -1$ we calculate the norming constants from

$$C_j(0) = b_j/a'(kj), j = 1, 2..., J,$$

where the terms $a'(kj), \bar{a}'(\bar{k}_j)$ are computed via the trace formulae [22]. Following the same procedure as in [22] we also find

$$b(k_j, -t)b(k_j, t) = 1 \Rightarrow b(k_j, 0) = \pm 1, \tag{235}$$

and

$$\bar{b}(k_j, -t)\bar{b}(k_j, t) = 1 \Rightarrow \bar{b}(k_j, 0) = \pm 1. \tag{236}$$

For a one-soliton solution, $\sigma = -1$, $J = 1$, the trace formulae yield

$$a'(k_1) = \frac{1}{k_1 - \bar{k}_1}, \bar{a}'(\bar{k}_1) = \frac{-1}{k_1 - \bar{k}_1} \Rightarrow \bar{a}'(\bar{k}_1) = -a'(k_1). \tag{237}$$

Thus,

$$C_1(0) = (k_1 - \bar{k}_1)b(k_1, 0), \bar{C}_1(0) = -(k_1 - \bar{k}_1)\bar{b}(k_1, 0).$$

This implies that

$$C_1^2(0) = \bar{C}_1^2(0).$$

The one soliton solution of the complex space-time nonlocal NLS equation (3) is again found using the above method with time evolution of the scattering data. We have

$$q(x,t) = \frac{2i\bar{C}_1(0)e^{-2i\bar{k}_1 x}e^{4i\bar{k}_1^2 t}}{1 + \frac{C_1(0)\bar{C}_1(0)}{(k_1 - \bar{k}_1)^2}e^{2i(k_1 - \bar{k}_1)x}e^{4i\left(\bar{k}_1^2 - k_1^2\right)t}}, \tag{238}$$

and

$$r(x,t) = -\frac{2iC_1(0)e^{2ik_1 x}e^{-4ik_1^2 t}}{1 + \frac{C_1(0)\bar{C}_1(0)}{(k_1 - \bar{k}_1)^2}e^{2i(k_1 - \bar{k}_1)x}e^{4i\left(\bar{k}_1^2 - k_1^2\right)t}}. \tag{239}$$

With $C_1^2(0) = \bar{C}_1^2(0)$ it follows that $r(x,t) = -q(-x,-t)$. Calling $k_1 = \xi_1 + i\eta_1, \bar{k_1} = \bar{\xi}_1 - i\bar{\eta}_1, \eta_1 > 0, \bar{\eta}_1 > 0$ and the above time dependence for $C_1(t), \bar{C}_1(t)$ leads to the one-soliton solution for Eq. (4)

$$q_{CSTNLS}(x,t) = \frac{2i\bar{C}_1(0)e^{-2i\bar{\xi}_1 x - 2\bar{\eta}_1 x}e^{4i\left(\bar{\xi}_1^2 - \bar{\eta}_1^2\right)t}e^{8\bar{\xi}_1\bar{\eta}_1 t}}{1 + \Gamma_1\Delta}, \tag{240}$$

where

$$\Delta = e^{-4i\left(\xi_1^2 - \eta_1^2\right)t + 4i\left(\bar{\xi}_1^2 - \bar{\eta}_1^2\right)t}e^{8\xi_1\eta_1 t + 8\bar{\xi}_1\bar{\eta}_1 t}e^{2i(\xi_1 - \bar{\xi}_1)x}e^{-2(\eta_1 + \bar{\eta}_1)x},$$

and $\Gamma_1 = C_1(0)\bar{C}_1(0)/[k_1 - \bar{k}_1]^2 = \gamma_1 = \pm 1$. The above soliton is stable in the sense that as $\bar{\xi}_1\bar{\eta}_1 \to \infty$ we find $q_{CSTNLS}(x,t)$ to be bounded. It also appears that if we let $\Gamma_1 = e^{2(\eta_1 + \bar{\eta}_1)x_0}e^{2i\psi_0}$ we can have a singularity when

$$-2(\eta_1 + \bar{\eta}_1)(x - x_0) + 8(\xi_1\eta_1 + \bar{\xi}_1\bar{\eta}_1)t = 0,$$

and

$$4\left(\left(\bar{\xi}_1^2-\bar{\eta}_1^2\right)-(\xi_1^2-\eta_1^2)\right)t+2\psi_0=(2n+1)\pi,\, n\in\mathbb{Z}.$$

The singularity can be eliminated by taking $(\bar{\xi}_1^2-\bar{\eta}_1^2)-(\xi_1^2-\eta_1^2)=0$ and $2\psi_0\neq(2n+1)\pi,\, n\in\mathbb{Z}$. As shown, the above symmetries yield solutions of NLS and nonlocal NLS type equations.

11.5. Complex reverse time symmetry: $r(x,t)=\sigma q^*(-x,-t)$

This symmetry yields a solution of the complex space-time nonlocal mKdV equation (3). The symmetries needed for this case are

$$a(k,t)=a^*(-k^*,-t)=a(k,0),$$

$$\bar{a}(k,t)=\bar{a}^*(-k^*,-t)=\bar{a}^*(-k^*,0),$$

$$\bar{b}(k,t)=\sigma b^*(-k,-t),\, k\in\mathbb{R}.$$

When $\sigma=-1$

$$k_1=i\eta,\, \eta>0,\, \bar{k}_1=-i\bar{\eta},\, \bar{\eta}>0, \tag{241}$$

$$C_1(t)=C_1(0)e^{8\eta^3t}, \tag{242}$$

$$\bar{C}_1(t)=\bar{C}_1(0)e^{8\bar{\eta}^3t}, \tag{243}$$

$$C_1(0)=i(\eta+\bar{\eta})b_1,\, b_1=e^{i(\theta+\pi)},\, \theta\in\mathbb{R},$$

$$\bar{C}_1(0)=-i(\eta+\bar{\eta})\bar{b}_1,\, \bar{b}_1=e^{i\bar{\theta}},\, \bar{\theta}\in\mathbb{R}.$$

Substituting into Eq. (209) yields the one-soliton solution of the complex nonlocal mKdV equation

$$q(x,t)=-\frac{2(\eta+\bar{\eta})e^{i\bar{\theta}}e^{-2\bar{\eta}x+8\bar{\eta}^3t}}{1+e^{i(\theta+\bar{\theta})}e^{-2\eta x+8\eta^3t-2\bar{\eta}x+8\bar{\eta}^3t}}. \tag{244}$$

We see that there are four real parameters in the above solution: $\eta,\bar{\eta},\theta,\bar{\theta}$. Another way to write this solution is as follows

$$q(x,t)=\frac{(\eta+\bar{\eta})e^{-i(\theta/2-\bar{\theta}/2)}e^{\eta(x-4\eta^2t)}e^{-\bar{\eta}(x-4\bar{\eta}^2t)}}{\cosh\left[(\eta(x-4\eta^2t)+\bar{\eta}(x-4\bar{\eta}^2t)-i(\theta+\bar{\theta})/2\right]}. \tag{245}$$

We see that this solution can be singular if $\theta+\bar{\theta}=(2n+1)\pi,\, n\in\mathbb{Z}$.

11.6. Real reverse space-time symmetry: $r(x,t) = \sigma q(-x,-t)$, $q \in \mathbb{R}$

There is only one change from the complex PT time reversal symmetry case,

$$C_1(0) = i(\eta + \bar{\eta})b_1, \bar{C}_1(0) = -i(\eta + \bar{\eta})\bar{b}_1, \tag{246}$$

but now with

$$b_1 = \pm 1, \bar{b}_1 = \pm 1.$$

Thus, the only difference from the complex PT time reversal symmetry case is that in the prior case we require $\theta, \bar{\theta} = 0, \pi$. Therefore, in this case there are only two free real parameters $\eta, \bar{\eta}$ and the real nonlocal mKdV equation (3) the one-soliton solution is given by

$$q(x,t) = \frac{2\gamma_1(\eta + \bar{\eta})e^{-2\bar{\eta}x + 8\bar{\eta}^3 t}}{1 + \gamma_2 e^{-2\eta x + 8\eta^3 t - 2\bar{\eta}x + 8\bar{\eta}^3 t}}, \tag{247}$$

where $\gamma_j = \pm 1$, $j = 1, 2$. If, say $\gamma_1 = \gamma_2 = 1$ then the solution can be written in the following form:

$$q(x,t) = \frac{(\eta + \bar{\eta})e^{\eta(x - 4\eta^2 t)}e^{-\bar{\eta}(x - 4\bar{\eta}^2 t)}}{\cosh\left[(\eta(x - 4\eta^2 t) + \bar{\eta}(x - 4\bar{\eta}^2)t\right]}. \tag{248}$$

This solution is not singular. When $\eta = \bar{\eta}$ the solution reduces to the well-known solution of the real mKdV equation

$$q(x,t) = \frac{2\eta}{\cosh\left[(2\eta(x - 4\eta^2 t)\right]}. \tag{249}$$

Finally, we construct soliton solution for the (real) space-time nonlocal sine-Gordon equation (82). The sG equation belongs to the same symmetry class as the space-time nonlocal mKdV equation. As such, for the one-soliton solution, the eigenvalues are given by $k_1 = i\eta_1$ and $\bar{k}_1 = -i\bar{\eta}_1$ with $\eta_1 > 0$ and $\bar{\eta}_1 > 0$. Furthermore, the evolution of the norming constants is given by Eqs. (206) and (207):

$$C_1^{\mathrm{sG}}(t) = C_1(0)e^{-t/(2\eta_1)}, \tag{250}$$

$$\overline{C}_1^{\mathrm{sG}}(t) = \overline{C}_1(0)e^{-t/(2\bar{\eta}_1)}. \tag{251}$$

The solution is thus found from Eq. (209) to be

$$q(x,t) = \frac{2ie^{-2\bar{\eta}_1 x}\overline{C}_1(0)e^{-t/(2\bar{\eta}_1)}}{1 - \dfrac{C_1(0)\bar{C}_1(0)e^{-t/(2\eta)}}{(\eta_1 + \bar{\eta}_1)^2 e^{2(\eta_1 + \bar{\eta}_1)x}}}, \tag{252}$$

where $C_1(0) = i(\eta + \bar{\eta})b_1$, $\bar{C}_1(0) = -i(\eta + \bar{\eta})\bar{b}_1$, $b_1 = \pm 1$, $\bar{b}_1 = \pm 1$ and

$$\frac{1}{\eta} = \frac{1}{\eta_1} + \frac{1}{\bar{\eta}_1}.$$

12. Conclusion and outlook

More than 40 years have passed since AKNS published their paper: "Inverse scattering transform—Fourier analysis for nonlinear problems," which appeared in this journal in 1974. Until recently, it was thought that all "simple" and physically relevant symmetry reductions of the "classical" AKNS scattering problem had been identified. However, in 2013, the authors discovered a new "hidden" reduction of the PT symmetric type which leads to a nonlocal NLS equation that admits a novel soliton solution. Surprisingly enough, the AKNS symmetry reduction found in [14] is not the end of the story. In this paper we unveil many new "hidden" symmetry reductions that are nonlocal both in space and time and, in some cases, nonlocal in time-only. Each new symmetry condition give rise to its own new nonlocal nonlinear integrable evolution equation. These include the reverse time NLS equation, reverse space-time nonlocal forms of the NLS equation, derivative NLS equation (which includes the reverse space-time nonlocal derivative NLS equation as a special case), loop soliton, modified Korteweg-deVries (mKdV), sine-Gordon, (1+1)- and (2+1)-dimensional multiwave/three-wave interaction, reverse discrete-time nonlocal discrete integrable NLS models and DS equations. Linear Lax pairs and an infinite number of conservation laws are discussed along with explicit soliton solutions in some cases. All equations arise from remarkably simple symmetry reductions of AKNS and related scattering problems. For convenience, below we list some of the symmetries associated with the AKNS scattering problem (21–24).

$$r(x,t) = \sigma q^*(x,t), \tag{253}$$

$$r(x,t) = \sigma q^*(-x,t), \tag{254}$$

$$r(x,t) = \sigma q(-x,-t), q \in \mathbb{C}, \tag{255}$$

$$r(x,t) = \sigma q(x,-t), q \in \mathbb{C}, \tag{256}$$

$$r(x,t) = \sigma q^*(-x,-t), \tag{257}$$

$$r(x,t) = \sigma q(-x,-t), q \in \mathbb{R}, \tag{258}$$

where $\sigma \mp 1$. In future work, these symmetries will be extended to other vector, matrix AKNS and $(2+1)$-dimensional AKNS-type systems. The symmetry (253) was discussed in [12] along with the subcase $r(x,t) = \sigma q(x,t), q \in \mathbb{R}$. The symmetry (254) was first discussed in [14], particularly with application to the PT symmetric NLS equation and related hierarchies. The symmetry (255) was first noted in [22] with regard to the nonlocal mKdV and SG equations, though the IST and one soliton solutions were not given there. We show here that the symmetries (253), (254), (255), and (256) are all associated with the IST and solutions of the NLS and nonlocal NLS equations while the symmetries (253, (257), and (258) are associated with the IST and solutions of the mKdV and nonlocal SG equation.

We close this section with an outlook toward future research direction pertaining to the emerging field of integrable nonlocal equations including what we here term here as *reverse space-time and reverse time systems*.

1. *IST and left-right Riemann-Hilbert (RH) problems for reverse space-time and inverse scattering for the reverse time-only nonlocal NLS type equations*.
   In [14, 22], it was shown that a "natural" approach to solve the inverse problem associated with the nonlocal NLS equation (2) is to formulate two separate RH problems: one for $x < 0$ (left) and one at $x > 0$ (right) then use the appropriate (nonlocal) symmetries between the eigenfunctions to reduce the number of independent equations and recover the potentials q and r. The left–right RH approach has the advantage of reducing the integral equations on the inverse side to integral equations for one function. It will be valuable to develop the left–right RH equations for the reverse space-time nonlocal equations and thereby develop a more complete inverse scattering theory. Indeed, inverse scattering is an important field of mathematics and physics independent of solving nonlinear equations.
2. *Nonlocal Painlevé type equations*. The Painlevé equations are certain class of nonlinear second-order complex ordinary differential equations that usually arise as reductions of the "soliton evolution equations," which are solvable by IST cf. [13]. They are particularly interesting due to their properties in the complex plane and their associated integrability properties. The first nonlocal (in space) Painlevé type equation was obtained in [14] and came out of a reduction of Eq. (2). Using the ansatz

$$q(x,t) = \frac{1}{(2t)^{1/2}} f(z) e^{i\nu \log t/2}, \qquad z = \frac{x}{(2t)^{1/2}}, \tag{259}$$

one can show that $f(z)$ satisfies

$$f_{zz}(z) + izf_z(z) + (\nu + i)f(z) - 2\sigma f^2(z)f^*(-z) = 0, \tag{260}$$

where $\sigma = \mp 1$. Since Eq. (260) comes out of Eq. (2) which, in turn arose using the so-called PT preserving symmetry reduction $r(x,t) = \sigma q^*(-x,t)$, we thus refer to (260) as a PT preserving Painlevé equation. The situation for the reverse space-time and reverse time only nonlocal NLS cases is different. Here, the proper ansatz we use for the reduction to ODE is of the form

$$q(x,t) = \frac{1}{(2t)^{1/2}} f(z), \qquad z = \frac{x}{(2t)^{1/2}}. \tag{261}$$

Substituting this ansatz into Eq. (3) gives

$$f_{zz}(z) + izf_z(z) + if(z) - 2\sigma\kappa f^2(z)f(\kappa z) = 0, \tag{262}$$

where $\sigma = \mp 1$ and $\kappa = (-1)^{-1/2}$. In this case, $\kappa = i$ if one chooses $-1 = e^{-i\pi}$ and $(-1)^{-1/2} = e^{i\pi/2}$ but $\kappa = -i$ if one chooses $-1 = e^{i\pi}$ and $(-1)^{-1/2} = e^{-i\pi/2}$, i.e., it is branch dependent. Since the number κ is branch dependent, it can wait to be defined when one does an application. On the other hand, from Eq. 4 one obtains the following ODE reduction

$$f_{zz}(z) + izf_z(z) + if(z) - 2\sigma\kappa f^2(z)f(-\kappa z) = 0, \tag{263}$$

with $\sigma = \mp 1$. Equations (262) and (263) are nonlocal Painlevé type equations. As a future research direction, it would be interesting to study the behavior of solutions to the above new nonlocal Painlevé equations.

3. *IST for the reverse time discrete and the reverse discrete-time NLS equation*. In Section 9, we used various discrete symmetry reductions based on the Ablowitz–Ladik scattering problem to obtain two new discrete nonlocal in both "space" and time nonlinear Schrödinger equation. A future research direction would be to develop the full IST and obtain soliton solutions of these equations.

Acknowledgments

The research of M.J.A. was partially supported by NSF under Grant No. DMS-1310200. We thank Dr. Xudan Luo for helpful interactions.

Appendix

In this Appendix, for the physical space symmetries discussed in this paper we will provide the symmetries associated with the AKNS eigenfunctions. To do so, we call $v(x,k) \equiv (v_1(x,k),\ v_2(x,k))^{\mathrm{T}}$ a solution to system (21). Note: $\sigma = \mp 1$.

1. For the standard AKNS symmetry (256), i.e., $r(x,t) = \sigma q^*(x,t)$ we have

$$\overline{\psi}(x,t,k) = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} \psi^*(x,t,k^*), \tag{A.1}$$

and

$$\overline{\phi}(x,t,k) = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} \phi^*(x,t,k^*). \tag{A.2}$$

2. For the reverse time AKNS symmetry (256), i.e., $r(x,t) = \sigma q(x,-t)$ we have

$$\overline{\psi}(x,t,k) = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} \psi(x,-t,k), \tag{A.3}$$

and

$$\overline{\phi}(x,t,k) = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} \phi(x,-t,k). \tag{A.4}$$

3. For the PT symmetry (254), i.e., $r(x,t) = \sigma q^*(-x,t)$ we have

$$\psi(x,t,k) = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix} \phi^*(-x,t,-k^*), \tag{A.5}$$

$$\overline{\psi}(x,t,k) = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix} \overline{\phi}^*(-x,t,-k^*). \tag{A.6}$$

4. For the reverse space-time symmetry (255), i.e., $r(x,t) = \sigma q(-x,-t), q \in \mathbb{C}$ we have

$$\psi(x,t,k) = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix} \phi(-x,-t,k), \tag{A.7}$$

$$\overline{\psi}(x,t,k) = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix} \overline{\phi}(-x,-t,k). \tag{A.8}$$

5. For the complex reverse space-time symmetry (255), i.e.,
$r(x,t) = \sigma q^*(-x,-t)$ we have

$$\psi(x,t,k) = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix} \phi^*(-x,-t,-k^*), \tag{A.9}$$

$$\overline{\psi}(x,t,k) = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix} \overline{\phi}^*(-x,-t,-k^*). \tag{A.10}$$

6. For the real reverse space-time symmetry (255), i.e.,
$r(x,t) = \sigma q(-x,-t), q \in \mathbb{R}$ we have the above symmetry given in item (5) associated with $r(x,t) = \sigma q^*(-x,-t)$ and

$$\psi(x,t,k) = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix} \phi(-x,-t,k), \tag{A.11}$$

$$\overline{\psi}(x,t,k) = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix} \overline{\phi}(-x,-t,k). \tag{A.12}$$

The above symmetry relations can be turned into symmetry relations for the scattering data $a(k), b(k)$ and eigenvalues $k_j, \bar{k}_j, j = 1, 2...J$ from the Wronskian relations (192), (193), (194), and (195). Finally symmetries for the normalization coefficients $C_j, \bar{C}_j, j = 1, 2...J$ can be found either directly from the above by analytic continuation or by individually finding b_j and $a'(k_j)$ associated with $C_j = b_j/a'(k_j)$ and $\bar{b}_j$ and $\bar{a}'(k_j)$ associated with $\bar{C}_j = \bar{b}_j/\bar{a}'(k_j)$ as was done in [22].

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(Received September 21, 2016)