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Analytical solutions to a class of nonlinear Schrödinger equations with $\mathcal{PT}$-like potentials

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Abstract
We present closed form solutions to a certain class of one- and two-dimensional nonlinear Schrödinger equations involving potentials with broken and unbroken $\mathcal{PT}$ symmetry. In the one-dimensional case, these solutions are given in terms of Jacobi elliptic functions, hyperbolic and trigonometric functions. Some of these solutions are possible even when the corresponding $\mathcal{PT}$-symmetric potentials have a zero threshold. In two-dimensions, hyperbolic secant type solutions are obtained for a nonlinear Schrödinger equation with a non-separable complex potential.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

One of the pillars of quantum mechanics is the Hermiticity of every operator associated with a physical observable. This is necessary given that the spectra of such self-adjoint operators—whether discrete or continuous—must always be real. In the case of the Hamiltonian operator, this requirement not only leads to real eigen energies but also ensures conservation of probability [1]. In the late nineties, the notion of Hermiticity has been critically re-examined by Bender and coworkers [2–5]. More specifically it was shown that non-Hermitian Hamiltonians exhibiting parity-time ($\mathcal{PT}$) symmetry could have entirely real spectra. A Hamiltonian is $\mathcal{PT}$ symmetric provided that it shares a common set of eigenfunctions with the $\hat{P}\hat{T}$ operator. In general the action of the parity operator $\hat{P}$ is defined by the relations $\hat{p} \to -\hat{p}, \hat{x} \to -\hat{x}$ ($\hat{p}, \hat{x}$ stand for momentum and position operators, respectively) whereas that of the time operator $\hat{T}$ by $\hat{p} \to -\hat{p}, \hat{x} \to \hat{x}, i \to -i$. Starting from the properties of the time operator $\hat{T}$ one can show that, $\hat{T}\hat{H} = \hat{p}^2/2 + V^*(x)$, and hence $\hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T} = \hat{p}^2/2 + V^*(-x) = \hat{H}$. As a result a Hamiltonian is $\mathcal{PT}$ symmetric as long as its potential has the following property: $V(x) = V^*(-x)$ [2–5]. Evidently, the real part of a $\mathcal{PT}$ complex potential must be a symmetric function of position whereas the imaginary component should be anti-symmetric. We emphasize that this latter requirement is necessary but not sufficient to guarantee the reality of the spectrum [2, 3]. In fact, one of the most fascinating features of such pseudo-Hermitian Hamiltonians is the existence of a sudden phase transition (after a critical threshold) beyond which the eigenvalues of the system become partly or entirely complex. This transition is better known in the literature as a spontaneous $\mathcal{PT}$ symmetry breaking [2, 3]. The potential of these recent mathematical developments in various areas of physics has also been discussed in a number of studies [2–10]. We would like to note that thus far most of the analytical studies in this field (where the closed form solutions are possible) have been carried out in the linear domain [11–13]. Over the years other non-Hermitian physical systems have also been studied both theoretically and experimentally [14–17].

Quite recently the possibility of realizing $\mathcal{PT}$-symmetric structures within the realm of optics has been suggested [18–20]. This was done by exploiting the mathematical correspondence between the quantum Schrödinger equation and the paraxial equation of diffraction. In these studies it was shown that optical $\mathcal{PT}$ synthetic materials can lead to altogether new behavior that is impossible in standard systems. Such effects include double refraction, power oscillations, eigenfunction unfolding and non-reciprocal diffraction patterns to mention a few [18, 19]. In addition, $\mathcal{PT}$ configurations are described by a unique algebra and hence much of their behavior (including their coupled mode analysis) has to be properly rederived [20]. The proposed optical $\mathcal{PT}$ systems can be realistically implemented through a judicious inclusion of gain/loss regions in guided wave geometries [18]. In the suggested optical analogy the complex refractive index distribution plays the role of the optical potential. The parity-time condition implies that the real index profile should be even in the transverse direction while the loss/gain distribution must be odd. Gain/loss levels of approximately $\pm 40\ \mathrm{cm}^{-1}$ at wavelengths of $\approx 1\ \mu$m, that are typically encountered in standard quantum well semiconductor lasers or semiconductor optical amplifiers and photorefractive crystals should be sufficient to observe $\mathcal{PT}$ behavior [21, 22]. Optical nonlinearities (quadratic, cubic, photorefractive, etc) provide an additional degree of freedom since they may allow one to study such configurations under nonlinear conditions [19, 22].

In this paper we present analytical solutions to a certain class of one- and two-dimensional nonlinear Schrödinger equations involving potentials with broken and unbroken $\mathcal{PT}$ symmetry. In the one-dimensional case, these solutions are given in terms of Jacobian elliptic functions, hyperbolic and trigonometric functions. Some of these solutions are possible even when the corresponding $\mathcal{PT}$-symmetric potentials have a zero threshold. In two-dimensions, hyperbolic secant type solutions are obtained for a nonlinear Schrödinger equation with a non-separable non-Hermitian potential.

2. The one-dimensional $\mathcal{PT}$-symmetric nonlinear Schrödinger equation

We begin our analysis by first considering one-dimensional optical wave propagation in a Kerr nonlinear system that involves a $\mathcal{PT}$-symmetric complex index distribution. In this case, the optical beam evolution is governed by the following normalized nonlinear Schrödinger (NLS)-like equation [18, 19],

$$\mathrm{i}\frac{\partial\psi}{\partial z} + \frac{\partial^2\psi}{\partial x^2} + [V(x) + \mathrm{i}W(x)]\psi + g|\psi|^2\psi = 0, \tag{1}$$

where ψ is proportional to the electric field envelope, z is a scaled propagation distance and $g = +1$ corresponds to a self-focusing nonlinearity while $g = -1$ to a defocusing one. In equation (1), optical diffraction is described by the ψ_{xx} term and the complex $\mathcal{PT}$-symmetric index distribution by the quantity $V(x) + iW(x)$. Based on the previous discussion, the real and the imaginary components of the $\mathcal{PT}$-symmetric potential satisfy the following relations: $V(-x) = V(x)$, $W(-x) = -W(x)$, respectively. Physically, $V(x)$ is associated with index guiding while $W(x)$ represents the gain/loss distribution of the optical potential.

Nonlinear stationary solutions to equation (1) are sought in the form

$$\psi(x, z) = \phi(x) \exp(\mathrm{i}\lambda z), \tag{2}$$

where $\phi(x)$ is the nonlinear eigenmode (which in general is a complex function) and λ is the corresponding real propagation constant. In this case ϕ satisfies

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + [V(x) + \mathrm{i}W(x)]\phi + g|\phi|^2\phi = \lambda\phi. \tag{3}$$

Equation (3) is supplemented with an appropriate set of boundary conditions. More specifically we consider: (i) localized solutions approaching zero at infinity and (ii) periodic solutions.

2.1. Self-focusing case

In order to investigate optical wave propagation in self-focusing media we set $g = +1$ in equation (1) (positive nonlinearity). As we will see, depending on the choice of optical potentials and boundary conditions, the nonlinear equation (3) can admit various types of exact solutions. As first demonstrated in [12], if the $\mathcal{PT}$ potential is of the Scarff II type, e.g.

$$V(x) = V_0 \sec h^2(x), \qquad W(x) = W_0 \sec h(x) \tanh(x), \tag{4}$$

then the nonlinear Schrödinger equation (3) admits an exact solution (corresponding to zero boundary conditions at $\pm\infty$) of the form

$$\phi = \phi_0 \sec h(x) \exp[\mathrm{i}\mu \tan^{-1}(\sinh(x))], \tag{5}$$

where $\mu = W_0/3$, $\lambda = 1$ and $\phi_0 = \sqrt{2 - V_0 + \left(W_0^2/9\right)}$. In equation (4), V_0 and W_0 are the amplitudes of the real and imaginary parts that satisfy $W_0 \leqslant V_0 + 1/4$ in order to ensure that the system is below the phase transition point [12]. In addition we have found that these modes are nonlinearly stable over a certain range of potential parameters [19]. As we will see, other classes of hyperbolic secant-like solutions can be obtained corresponding to different types of localized $\mathcal{PT}$ potentials.

In what follows we show that equation (3) also admits a family of periodic solutions residing on a $\mathcal{PT}$-like complex periodic lattice [23–25]. To demonstrate such states, we consider the following Jacobian periodic potentials:

$$V(x) = V_0 sn^2(x, k) + W_0^2 k^2 sn^4(x, k), \tag{6}$$

$$W(x) = W_0 sn(x, k)[4\mathrm{d}n^2(x, k) - k'^2] \tag{7}$$

where $sn(x, k)$ denotes the Jacobi elliptic function with elliptic modulus $0 \leqslant k \leqslant 1$ and $k'^2 = 1 - k^2$ is the complementary elliptic modulus. We here assume that$W_0 \neq 0$. Since $sn(x, k)$ and $\mathrm{d}n(x, k)$ are periodic in x with period $4K(k)$ and $2K(k)$, respectively, with $K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\zeta}{\sqrt{1-k^2 \sin^2 \zeta}}$ being a complete elliptic integral, then $V(x)$ and $W(x)$ are periodic functions with period $2K(k)$ and $4K(k)$ respectively (for $0 \leqslant k < 1$). To better understand the structure of the potential $V(x)$, we examine the location of its critical points that are given as solutions to $sn(x, k)cn(x, k)\left[V_0 + 2W_0^2 k^2 sn^2(x, k)\right] = 0$. When V_0 is strictly positive, we find that over one period of the potential $V(x)$, the critical points 0, $2K(k)$ are local minima with a minimum value of $V(x)$ being zero. On the other hand, the critical point $x = K(k)$ is a

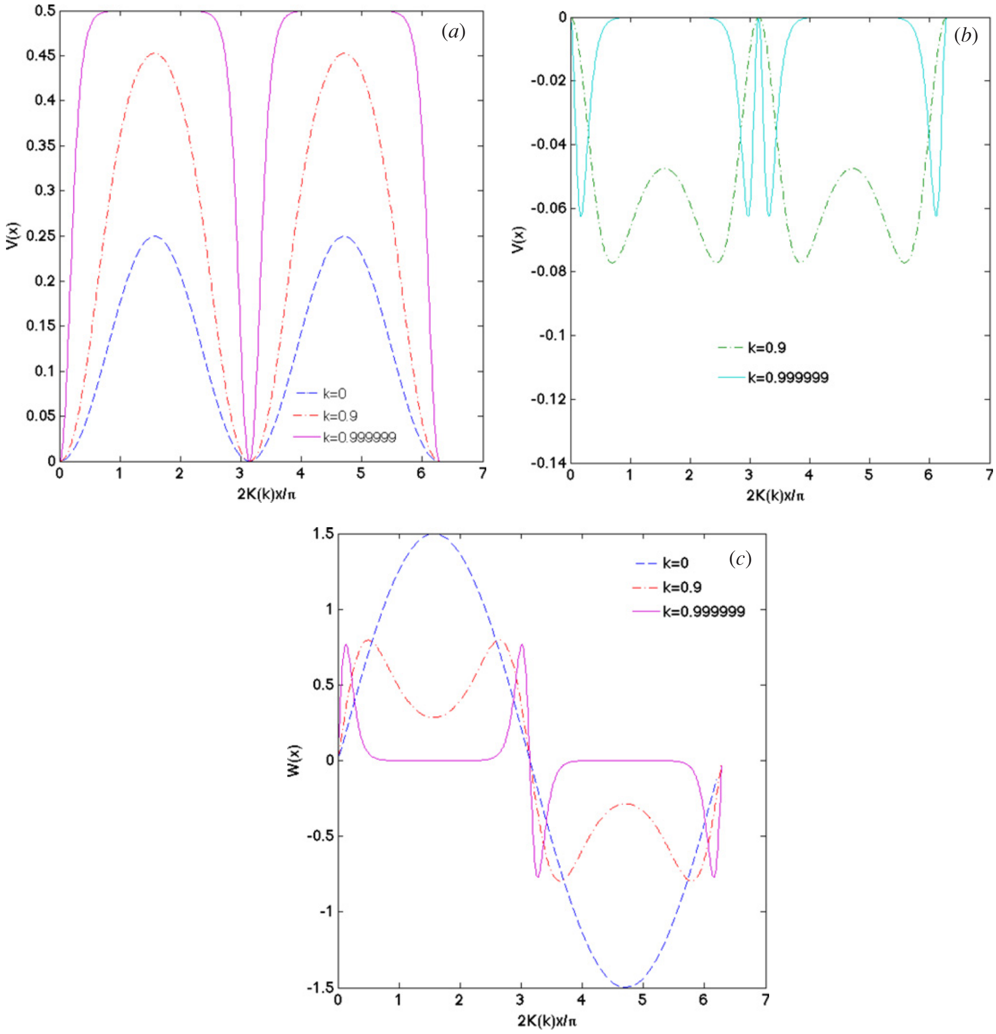

Figure 1. (*a*) The structure of the potential $V(x)$ given in equation (6) for various values of k. Note that the x coordinate has been scaled by the period of the elliptic function. Parameters are: $V_0 = 0.25$ and $W_0 = 0.5$. (*b*) The same as (*a*) except for parameters $V_0 = -0.25$ and $W_0 = 0.5$ with $k_c = 0.7071$. (*c*) The structure of the potential $W(x)$ given in equation (7) for various values of k and $W_0 = 0.5$. Note that the x coordinate has been scaled by the period of the elliptic function.

local maximum whose value is equal to $V_0 + W_0^2k^2$. In figure 1(*a*) we show a typical behavior of $V(x)$ for $V_0 = 0.25$ and $W_0 = 0.5$ for various values of the elliptic modulus k.

The situation can be quite different when $V_0 \leqslant 0$. Here, an additional critical point(s) (one or two) can appear due to a contribution from the term in the square bracket. Since $sn(x, k)$ is bounded from above by 1, the extra critical point(s) exist if $|\tilde{V}_0| < \sqrt{2}|W_0|$, and $k > k_c \equiv |\tilde{V}_0|/\sqrt{2}|W_0|$, where we have defined $V_0 \equiv -\tilde{V}_0^2$. In figure 1(*b*) we show a typical behavior of the potential $V(x)$ for $V_0 = -0.25$ and $W_0 = 0.5$ with $k_c = 0.7071$. Note that when $k = 0.7$, no extra fixed points show up, whereas two additional fixed points appear within one

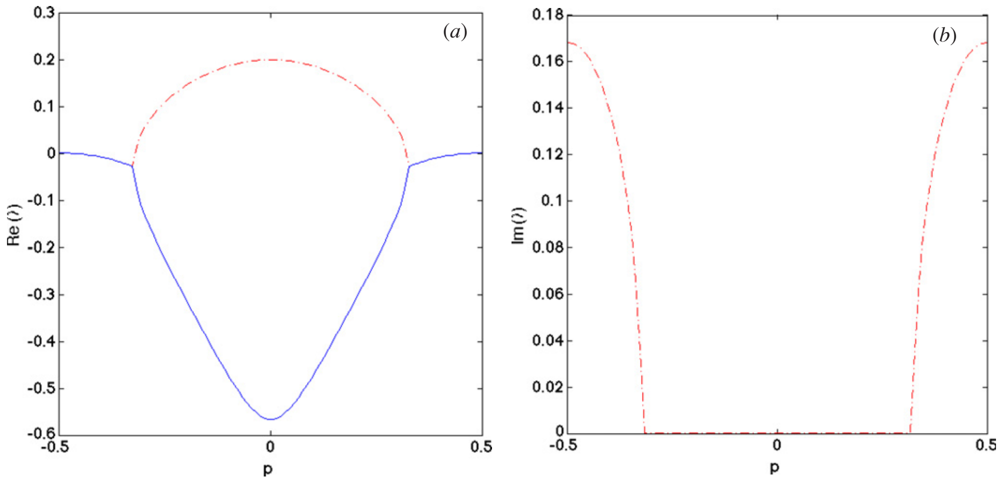

Figure 2. (*a*) Real part of the band structure for the potentials in equations (6) and (7) obtained for potential parameters $V_0 = 0.5$, $W_0 = 0.1$ and elliptic modulus $k = 0.1$. (*b*) Imaginary part of the band structure for the potentials in equations (6) and (7) obtained for potentials parameters $V_0 = 0.5$, $W_0 = 0.1$ and elliptic modulus $k = 0.1$.

period of the potential for higher values of k. Also, in figure 1(*c*), we show a typical sketch of the potential $W(x)$ for $W_0 = 0.5$ for various values of the elliptic modulus k.

Next, we examine the following linear eigenvalue problem:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + [V(x) + \mathrm{i}W(x)]\phi = \lambda\phi \tag{8}$$

that corresponds to the linearized version of equation (3). This will allow us to determine the parameter range for which the eigenvalues λ are real. To do so, we first take advantage of the fact that the combined potential $V(x) + \mathrm{i}W(x)$ is a periodic function of x which according to the Floquet–Bloch theorem implies that the eigenfunctions ϕ satisfy $\phi = \Phi_p(x)\exp(\mathrm{i}px)$, $\Phi_p(x + 4K(k)) = \Phi_p(x)$, where p stands for the real Bloch momentum. In this paper, we will consider nonlinear periodic solutions with zero Bloch momenta ($p = 0$). We note that in general the band structure, $\lambda(p)$, of a non-Hermitian lattice can be complex. Yet, for periodic $\mathcal{PT}$-symmetric potentials, the band diagram can be entirely real as long as the potentials have unbroken $\mathcal{PT}$ symmetry. The eigenvalue problem (8) was solved numerically using spectral methods and was found that the spectrum $\lambda(p)$ is complex for all values of V_0, W_0 and k. This implies that the potentials of equations (6) and (7) have a zero threshold point, i.e., the $\mathcal{PT}$ symmetry is always broken for any finite W_0. In figure 2 we show a typical band structure (real and imaginary) corresponding to the potentials in equations (6) and (7) for $V_0 = 0.5$, $W_0 = 0.1$ and $k = 0.1$. To intuitively understand why in this case the spectrum can be partly complex, we consider a simplified discrete (tight-binding) model based on a diatomic chain (the potentials given in (6) and (7) resemble a diatomic chain). In the tight-binding model the equations describing wave propagation in a linearly coupled diatomic chain are governed by [22]

$$\mathrm{i}\frac{\mathrm{d}a_n}{\mathrm{d}z} + \mathrm{i}\gamma a_n + b_n + b_{n-1} = 0, \tag{9}$$

$$\mathrm{i}\frac{\mathrm{d}b_n}{\mathrm{d}z} - \mathrm{i}\gamma b_n + a_n + a_{n+1} = 0, \tag{10}$$

where γ is the gain/loss coefficient. We then assume a plane wave solution (a discrete Floquet–Bloch mode) of the form

$$a_n = A\,\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{e}^{\mathrm{i}Qn}, \tag{11}$$

$$b_n = B\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{e}^{\mathrm{i}Qn}, \tag{12}$$

where ω is the longitudinal wavenumber (spatial eigen energy) and Q is the Bloch momentum that lies within the reduced Brillouin zone $[-\pi/2, \pi/2]$. Substituting equations (11) and (12) into equations (9) and (10) we find that a nontrivial solution exists only if the wavenumber ω satisfies the following:

$$\omega^2 = 4\cos^2(Q/2) - \gamma^2. \tag{13}$$

It is evident from the dispersion relation of equation (13), that at the edge of the Brillouin zone the spectrum can indeed become complex for any finite gain/loss coefficient γ. Yet the fact that the linear spectrum is not entirely real does not exclude the possibility of finding nonlinear waves (solutions to equation (3)) that propagate without any change in intensity. Indeed for the potentials given in equations (6) and (7) a periodic solution to equation (3) is given by

$$\phi = \phi_0\, cn(x,k)\exp[\mathrm{i}\theta(x)], \tag{14}$$

where

$$\phi_0 = \sqrt{V_0 + W_0^2k^2 + W_0^2 + 2k^2}, \tag{15a}$$

$$\lambda = V_0 + W_0^2k^2 + 2k^2 - 1 \tag{15b}$$

and

$$\theta(x) = W_0 sn(x,k). \tag{16}$$

The solution exists in the branch $V_0 > -\left(W_0^2k^2 + W_0^2 + 2k^2\right)$. Note that the amplitude of this solution is an even function of x whereas the phase is odd. In figure 3 we show a typical profile of the real and imaginary parts of the nonlinear eigenmode (equation (14)) together with the potentials given by equations (6) and (7). The case $k = 0$ is particularly interesting since $sn(x,0) = \sin(x)$ and $cn(x,0) = \cos(x)$ and $dn(x,0) = 1$. This limit leads to a potential of the form

$$V(x) = V_0\sin^2(x), \qquad W(x) = 3W_0\sin(x). \tag{17}$$

In this same limit and for the complex potential of equations (17), from equations (14)–(16) one directly obtains a sinusoidal nonlinear 'Floquet–Bloch function' for the nonlinear Schrödinger equation, e.g.

$$\phi = \sqrt{V_0 + W_0^2}\cos(x)\exp[\mathrm{i}W_0\sin(x)], \tag{18}$$

that is valid for $V_0 > -W_0^2$ with $\lambda = V_0 - 1$ (see figure 4). The other interesting case occurs when the elliptic modulus is equal to unity ($k \to 1$). In this particular limit the Jacobian elliptic functions reduce to $cn(x,1) = \mathrm{d}n(x,1) = \sec h(x)$, $sn(x,1) = \tanh(x)$ and the potentials appearing in equations (6) and (7) take the form

$$V(x) = -\left(V_0 + 2W_0^2\right)\sec h^2(x) + W_0^2\sec h^4(x), \tag{19}$$

$$W(x) = 4W_0\tanh(x)\sec h^2(x), \tag{20}$$

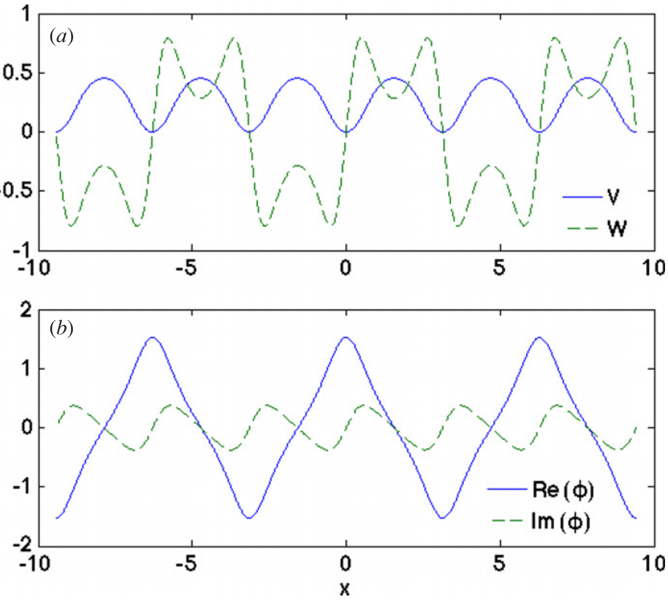

Figure 3. (*a*) The real and imaginary parts of the complex potential (equations (6) and (7)). (*b*) The real and imaginary components of the solution ϕ given by equation (14). Parameters are: $V_0 = 0.25$, $W_0 = 0.5$ and $k = 0.9$.

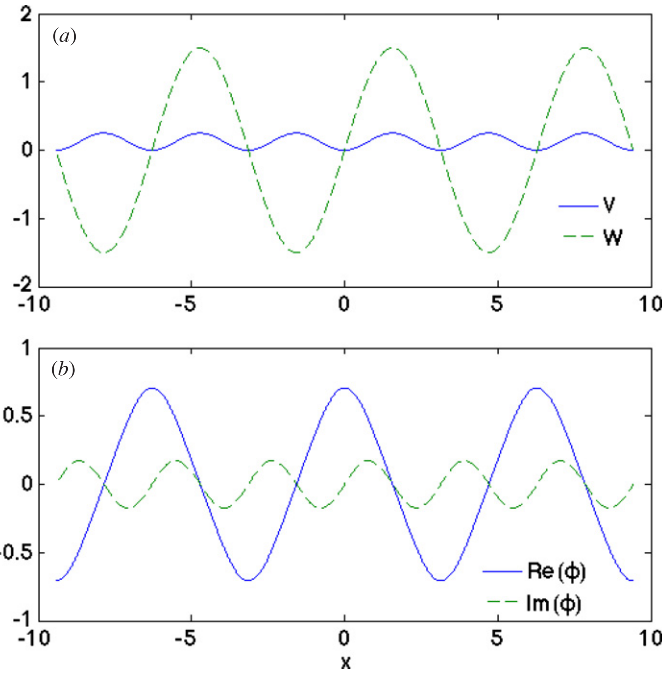

Figure 4. (*a*) The real and imaginary parts of the complex potential given by equation (17). (*b*) The real and imaginary components of the solution ϕ given by equation (18). Parameters are: $V_0 = 0.25$ and $W_0 = 0.5$.

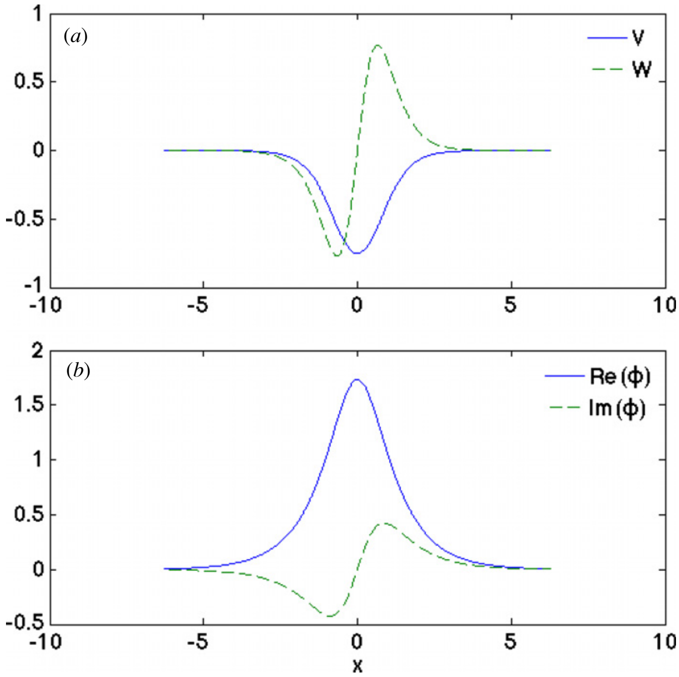

Figure 5. (*a*) The real and imaginary parts of the complex potential given by equations (19) and (20). (*b*) The real and imaginary components of the solution ϕ given by equation (21). Parameters are: $V_0 = 0.5$ and $W_0 = 0.5$.

where we have subtracted the constant term $V_0 + W_0^2$ from the potential in equation (19) and absorbed it in the definition of the eigenvalue λ (see equation (15)). It turns out that the potentials in (19) and (20) exhibit a nonzero threshold for unbroken $\mathcal{PT}$ symmetry. In this case the solution corresponding to the potential of equations (19) and (20) is thus given by

$$\phi = \sqrt{V_0 + 2W_0^2 + 2}\sec h(x)\exp[\mathrm{i}W_0\tanh(x)], \tag{21}$$

with $V_0 > -2\left(W_0^2+1\right)$ and $\lambda = 1$. In figure 5 we show a graph of both potentials together with the wavefunction (21) corresponding to the parameters value $V_0 = W_0 = 1/2$. Interestingly enough, if we assume $V_0 = -W_0^2$ then the potential appearing in equation (19) becomes $V(x) = -W_0^2\tanh^2(x)\sec h^2(x)$ while the imaginary potential of equation (14) remains the same. In this latter case the nonlinear wave solution is still of the hyperbolic secant type (equation (21)) with an amplitude $\phi_0 = \sqrt{W_0^2 + 2}$.

Before ending this section, we would like to note that equation (1) also admits another type of elliptic function solutions corresponding to other forms of potentials. For example, solutions to equation (3) corresponding to the following periodic potentials:

$$V(x) = \left[V_0 + W_0^2 cn^2(x,k)\right]\mathrm{d}n^2(x,k), \tag{22}$$

$$W(x) = W_0 sn(x,k)[4\,\mathrm{d}n^2(x,k) - 3k'^2], \tag{23}$$

are given by

$$\phi = \sqrt{2 - V_0}\,\mathrm{d}n(x,k)\exp[\mathrm{i}W_0 sn(x,k)], \tag{24}$$

valid for $V_0 < 2$ with $\lambda = 2 - k^2$. In the limiting case $k \to 0$ the potentials in equations (22) and (23) read $V(x) = V_0 + W_0^2 \cos^2(x)$ and $W(x) = W_0 \sin(x)$ which result in

$$\phi = \sqrt{2 - V_0} \exp[\mathrm{i} W_0 \sin(x)], \tag{25}$$

with $\lambda = 2$.

Another possibility arises when the elliptic modulus approaches unity ($k \to 1$). In that case we find that $V(x) = V_0 \sec h^2(x) + W_0^2 \sec h^4(x)$, $W(x) = 4W_0 \sec h^2(x) \tanh(x)$ and

$$\phi = \sqrt{2 - V_0} \sec h(x) \exp[\mathrm{i} W_0 \tanh(x)]. \tag{26}$$

We would like to emphasize that the periodic solutions found here in $\mathcal{PT}$-like lattices are to some extent generalizations of those previously reported in Kerr systems with real periodic potentials [26–28].

2.2. Self-defocusing case

Optical beam propagation in nonlinear self-defocusing Kerr media is governed by equation (1) with $g = -1$ and its corresponding stationary solutions obey equation (3). As in the self-focusing case, here periodic elliptic solutions also exist and have a similar structure to those found in section 2.1. To this end, we consider the following elliptic potentials:

$$V(x) = -V_0^2 sn^2(x, k) + W_0^2 k^2 sn^4(x, k), \tag{27}$$

$$W(x) = W_0 sn(x, k)[4\,\mathrm{d}n^2(x, k) - k'^2]. \tag{28}$$

Using the Floquet–Bloch theorem we were able to numerically construct the band diagram associated with the above potentials which happens to be similar to figure 2. We found that the potentials given in equations (27) and (28) exhibit again a zero threshold point for $\mathcal{PT}$ symmetry breaking. Even though the linear spectrum is not entirely real, a nonlinear eigenmode can still exist. Indeed, the solution corresponding to this latter defocusing case for the specific type of elliptic potentials (27) and (28) is given by

$$\phi = \sqrt{V_0^2 - W_0^2 k^2 - W_0^2 - 2k^2 cn(x, k)} \exp[\mathrm{i} W_0 sn(x, k)], \tag{29}$$

with $\lambda = W_0^2 k^2 + 2k^2 - V_0^2 - 1$valid for $V_0^2 > W_0^2 k^2 + W_0^2 + 2k^2$. The limiting cases $k \to 0, 1$ can be derived in a similar fashion as was done in section 2.1. Note that the solutions in the defocusing case are similar to those in the focusing regime except for their amplitudes and their domain of existence.

3. The two-dimensional $\mathcal{PT}$-symmetric nonlinear Schrödinger equation

In the absence of an external potential, the one-dimensional NLS equation is an integrable partial differential equation known to admit exact soliton solutions. However, in the presence of an external potential, the NLS equation is only known to allow closed form solutions in limited cases. Some of these results have quite recently been reported in conjunction with $\mathcal{PT}$-symmetric potentials [19].

The situation in two-dimensions (2D) is completely different. Here, the 2D NLS equation is not integrable. In this section, we present closed form solutions to the 2D NLS equation in the presence of a certain type of external $\mathcal{PT}$-symmetric potentials.

In the 2D case, equation (1) takes the form

$$\mathrm{i}\frac{\partial \psi}{\partial z} + \nabla^2 \psi + [V + \mathrm{i}W]\psi + |\psi|^2 \psi = 0, \tag{30}$$

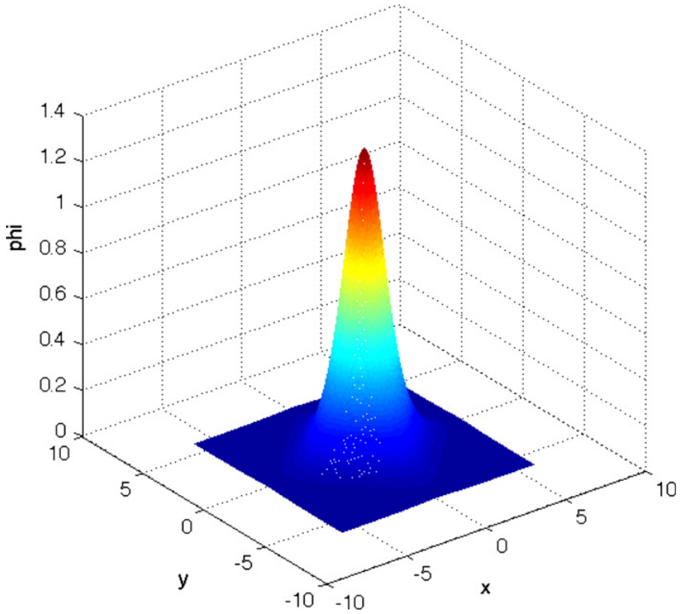

Figure 6. Plot of the eigenfunction ϕ for potential parameters $V_0 = 1$ and $W_0 = 0.5$.

where ∇^2 is the two-dimensional Laplacian and again the potentials V and W obey the $\mathcal{PT}$ requirement, $V(-x,-y) = V(x,y)$ and $W(-x,-y) = -W(x,y)$. Here we consider the complex potential

$$V = \left(2 + W_0^2/9\right)[\sec h^2(x) + \sec h^2(y)] + \left(V_0^2 - W_0^2/9 - 2\right)\sec h^2(x)\sec h^2(y), \tag{31}$$

$$W = W_0[\tanh(x)\sec h(x) + \tanh(y)\sec h(y)]. \tag{32}$$

Note that the real part of the potential V is not separable. Stationary nonlinear solutions are then sought in the form

$$\psi(x,y,z) = \phi(x,y)\exp[\mathrm{i}\lambda z + \mathrm{i}\theta(x,y)], \tag{33}$$

where ϕ and θ are real valued functions that satisfy the following differential equations:

$$\nabla^2\phi - |\nabla\theta|^2\phi + V(x,y)\phi + \phi^3 = \lambda\phi, \tag{34}$$

$$\phi\nabla^2\theta + 2\nabla\theta\cdot\nabla\phi + W(x,y)\phi = 0. \tag{35}$$

A bound state solution to equations (34) and (35) that satisfies the condition: $\phi \to 0$ as $(x,y) \to \pm\infty$ is given by

$$\phi(x,y) = \sqrt{2 - V_0^2 + \left(W_0^2/9\right)}\sec h(x)\sec h(y), \tag{36}$$

$$\theta(x,y) = \frac{W_0}{3}[\arctan(\sinh(x)) + \arctan(\sinh(y))], \tag{37}$$

with the propagation constant $\lambda = 2$. In figures 6 and 7 we show a typical profile of the eigenfunction ϕ and the phase θ for potential parameters $V_0 = 1$ and $W_0 = 0.5$.

It is interesting to note that for the real potential case ($W_0 = 0$) then $\theta = 0$ and equation (34) reduces to the classical 2D NLS equation with an external potential whose solution is given by equation (36) with $W_0 = 0$.

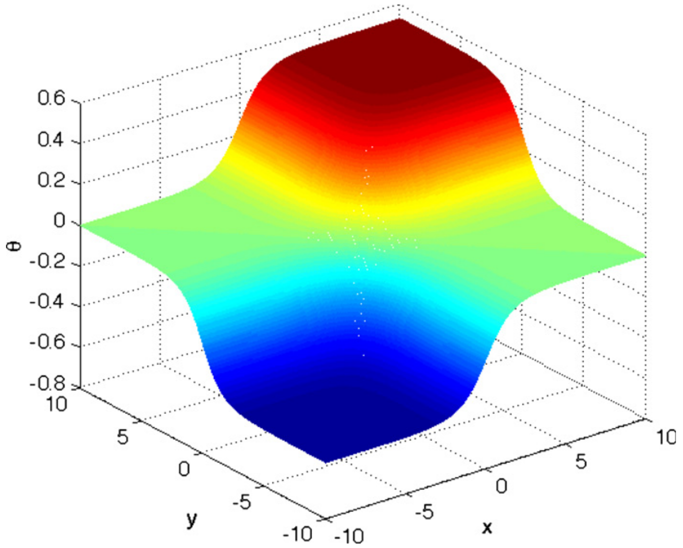

Figure 7. Plot of the phase θ for potential parameter $W_0 = 0.5$.

4. Conclusions

In conclusion, a new class of one- and two-dimensional nonlinear modes residing in parity-time symmetric wells and lattices is reported. In the one-dimensional case, it is shown that the solutions are of the Jacobi elliptic type. It is interesting to note that these solutions exist even though the $\mathcal{PT}$-like potentials have a zero threshold point (a point beyond which all the spectra of the potentials become partly complex). In the two-dimensional case, hyperbolic-secant type solutions are reported for both the standard and the $\mathcal{PT}$-symmetric nonlinear Schrödinger equation. Before closing we would like to mention that some issues related to this new class of waves may still merit further investigation. These include their stability analysis and its relation to the underlying band structure (in the case of a $\mathcal{PT}$-like lattice).

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