



# Generalized uncertainty principle and analogue of quantum gravity in optics



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## HIGHLIGHTS

- Analogy between quantum gravity and linear and nonlinear optics.
- We predict the existence of maximally localized states in nonlinear optics.
- The technique used overcomes the limits imposed by standard Fourier optics.
- We demonstrated that ideas from quantum gravity have relevance in nonlinear physics.

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## ABSTRACT

The design of optical systems capable of processing and manipulating ultra-short pulses and ultra-focused beams is highly challenging with far reaching fundamental technological applications. One key obstacle routinely encountered while implementing sub-wavelength optical schemes is how to overcome the limitations set by standard Fourier optics. A strategy to overcome these difficulties is to utilize the concept of a *generalized uncertainty principle* (G-UP) which has been originally developed to study quantum gravity. In this paper we propose to use the concept of G-UP within the framework of optics to show that the generalized Schrödinger equation describing short pulses and ultra-focused beams predicts the existence of a minimal spatial or temporal scale which in turn implies the existence of maximally localized states. Using a Gaussian wavepacket with complex phase, we derive the corresponding generalized uncertainty relation and its maximally localized states. Furthermore, we numerically show that the presence of nonlinearity helps the system to reach its maximal localization. Our results may trigger further theoretical and experimental tests for practical applications and analogues of fundamental physical theories.

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## 1. Introduction

For a given optical system such as a fiber or an imaging apparatus, understanding the shortest achievable pulse or the thinnest producible light spot is an issue of paramount importance for a large number of practical applications and fundamental sciences. In this regard, Fourier optics is the reference paradigm for designing ultrafast temporal processes, and imaging systems [1]. In

Fourier optics the uncertainty principle relates the spectral content of a beam to its spatial size thus allowing one to engineer optical systems and their numerical aperture for specific applications. However, the formalism of Fourier optics cannot be used for beams with size comparable to their wavelength because of the onset of nonparaxial effects.

Recent developments in the area of super-resolved microscopy [2], involve light beams with size much smaller than the wavelength in which case the standard Heisenberg uncertainty principle (H-UP) breaks down. Seemingly in the temporal domain, the uncertainty principle intervenes in determining the minimal duration for transform limited pulses [3]. However for ultra-short pulses [3], higher order dispersion fails to predict the shortest accessible signal with the use of simple Fourier optics.

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To generalize the uncertainty principle to tackle the challenge of determining the smallest possible beam or the shortest optical pulse for a given spatial and temporal dispersion, there is the need of looking at novel techniques. In the following we show that unexpectedly quantum gravity furnishes a possible road.

Many quantum gravity models predict a space discretization which results in having a minimal uncertainty length  $\Delta x_{\min}$ . This feature is inferred by a modification of the standard uncertainty principle of quantum mechanics to a generalized uncertainty principle which in the simplest form can be written as

$$\Delta x \Delta P > \frac{\hbar}{2} [1 + \beta (\Delta P)^2], \quad (1)$$

where  $\Delta P$  is the momentum uncertainty and  $\beta > 0$  is a parameter that takes into account the deviation from the standard Heisenberg uncertainty principle. The possible validity of a G-UP has been studied for decades as the key to solve fundamental problems in physics such as the transplanckian problem of the Hawking radiation, the modification of the blackbody radiation spectrum, corrections to cosmological constants and to the black-hole entropy [4,5].

Despite all these investigations, the value of  $\beta$  is unknown and its particular expression in terms of other physical constants, such as, the Planck length, varies depending on the various quantum gravity theories. It is often expressed in terms of the dimensionless parameter  $\beta_0 = M_p^2 c^2 \beta$ , with  $M_p$  being the Planck mass, and  $c$  is the speed of light in vacuum. Letting  $G$  denote the gravitational constant, and  $M_p = \sqrt{\hbar c / G}$  the Planck mass,  $\beta_0$  is also written as

$$\beta_0 = \frac{\hbar c^3}{G} \beta. \quad (2)$$

Some authors affirm that  $\beta_0 \cong 1$ , but a recent analysis poses the limit  $\beta_0 < 10^{34}$  [6,7]. Even in the case  $\beta_0 \cong 10^{34}$ , accessing experimentally measurable effects of a G-UP appears to be prohibitively difficult. In this regard, finding analogies in other branches of physical sciences would be very important since it could serve as a test bed for the newly reported G-UP predictions as well as providing insights for further theoretical developments and novel experiments.

There is an unexpected “link” between quantum gravity and nonparaxial and ultrafast optics [8]. The key point is that the first order non-paraxial theory (and seemingly the theory of pulse propagation with higher order dispersion) is formally identical to the modified quantum Schrödinger equation that is studied in the G-UP literature [6]:

$$i\hbar \partial_t \psi = \frac{\hat{p}^2}{2m} \psi + \frac{\beta}{3m} \hat{p}^4 \psi, \quad (3)$$

with  $\hat{p} = -i\hbar \partial_x$  being the quantum momentum,  $\psi$  is the quantum wave-function and  $m$  the particle mass. This mathematical analogy allows one to describe and test nonparaxial and ultrafast regimes for optical propagation in terms of the paradigms developed in the G-UP framework. As we detail below, in the optical analogues the values of  $\beta$  are such that one can foresee doable emulations of the physics at the Planck scale.

In this paper, we develop the concept of *generalized uncertainty principle* (G-UP) in the framework of linear and nonlinear optics. The generalized linear and nonlinear Schrödinger equations describing short pulses and ultra-focused beams are used to predict the existence of a minimal spatial or temporal scale. As a result, maximally localized states exist and their properties are discussed. The theoretical results are tested for a Gaussian wavepacket with complex phase. An explicit inequality for the generalized uncertainty relation is derived along with its corresponding maximally localized modes. We numerically show that the presence of nonlinearity helps the system to reach its maximally localized state.

The manuscript is organized as follows: in Section 2, we propose the higher order nonparaxial optical wave equation and show that it is formally equivalent to the generalized quantum Schrödinger equation (3) both in the temporal and spatial domains. We derive an explicit expression for the  $\beta$  parameter valid for optical settings. In Section 3, we find an expression for the G-UP in optics, derive the minimal uncertainty length,  $\Delta x_{\min}$ , and analyze its properties in the case of a chirped Gaussian wavepacket. In Section 4, we introduce and evaluate the maximally localized states, which are the states satisfying the G-UP strictly. As a final part, in Section 5, we show that these maximally localized states naturally occur in the nonlinear regime. Conclusions are drawn in Section 6.

## 2. Higher order nonparaxial wave equation

### 2.1. Spatial optics

We start this section by showing how the wave equation can be formally “mapped” to the quantum Schrödinger equation (3). To this end, we consider a unidimensional Helmholtz equation for the electric field  $\mathcal{E}$  and propagation direction  $z$

$$\partial_z^2 \mathcal{E} + \partial_x^2 \mathcal{E} + k_0^2 \mathcal{E} = 0, \quad (4)$$

where  $k_0 = 2\pi/\lambda$  with  $\lambda$  being the wavelength. We remark that vectorial effects are not present in vacuum [9–11]. Eq. (4) admits forward and backward propagating waves with longitudinal (i.e., in the  $z$ -direction) wavenumber

$$k_z = \pm \sqrt{k_0^2 - k^2}, \quad (5)$$

with  $k$  being the transverse wavenumber. Retaining only forward propagating beams, the forward projected Helmholtz equation (FPHE) reads [12]

$$i\partial_z \mathcal{E} + \sqrt{\partial_x^2 + k_0^2} \mathcal{E} = 0. \quad (6)$$

In general, the dispersion relation (5) describes both spatially periodic as well as evanescent waves. However, in this paper, we shall consider only dynamics of narrowly localized beams (in momentum space) corresponding to Fourier mode  $k$  satisfying the condition  $|k| \ll k_0$ . With this in mind, we expand the dispersion relation (5) in powers of  $k^2$  and obtain (retaining terms up to order  $k^4$ ) the first-order non-paraxial equation [13]

$$i\partial_z \mathcal{A} = -\frac{1}{2k_0} \partial_x^2 \mathcal{A} + \frac{1}{8k_0^3} \partial_x^4 \mathcal{A}, \quad (7)$$

with  $\mathcal{A} = \mathcal{E} e^{-ik_0 z}$ . To further establish the connection between G-UP in quantum mechanics and its optical analogue, we identify the value of the parameters  $\beta$  and  $\beta_0$ . Letting  $\partial_x = -\frac{i\hat{p}}{\hbar}$  and  $z = ct$  one obtains the following expression for the  $\beta$  parameter [8]

$$\beta = \frac{3}{8} \left( \frac{\lambda}{\hbar} \right)^2. \quad (8)$$

The formal (mathematical) identity between the unidirectional FPHE and the quantum Schrödinger equation allows one to provide an expression for the parameter  $\beta$  given in Eq. (8) and hence of its corresponding normalized  $\beta_0$ . In the optical case, from Eq. (2) and (8),  $\beta_0$  can be written as:

$$\beta_0 = \frac{3 M_p^2}{8 m^2} = \frac{3 c^3 (\lambda/2\pi)^2}{8 G \hbar}. \quad (9)$$

Table 1 shows typical values of  $\beta_0$  obtained from Eq. (9). In [6] it has been estimated  $\beta_0 < 10^{34}$ . We hence observe that, in the optical

**Table 1**

$\beta_0$  calculated from Eq. (9), for the neutron with  $v \cong c$ . Note that  $\beta_0 \simeq 1$  in the quantum gravity literature.

	$\lambda$ (m)	$m$ (kg)	$\beta_0$
Photon	$10^{-6}$	$10^{-36}$	$10^{55}$
$\gamma$ ray	$10^{-12}$	$10^{-31}$	$10^{45}$
Neutron	$10^{-15}$	$10^{-27}$	$10^{39}$

regime, G-UP effects for the photon are expected to be much more pronounced being  $\beta_0 = 10^{55}$ .

Quantum gravity effects are often considered to be unobservables, even if some possibilities have been reported in literature [6,7] but also questioned [14]. In our analogue, one can see that nonparaxial regimes for light allows to test some concepts introduced in the G-UP literature. In the same perspective, mathematical tools developed in the G-UP framework furnish novel roads for nonparaxial and ultrafast light propagation.

## 2.2. Temporal case

The formal analogy established in the spatial case can be also extended to the temporal domain for which the dynamics of a highly dispersive pulse is governed by [3]

$$i \frac{\partial A}{\partial z} - \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} - \frac{\beta_4}{4!} \frac{\partial^4 A}{\partial t^4} = 0, \quad (10)$$

where  $A$  is the pulse envelope function,  $t$  is the temporal variable and  $z$  is the propagation distance. We consider the case of a dispersion-flattened fiber with zero third order dispersion ( $\beta_3 = 0$ ) [3]. By defining the following rescaled variables  $z = Tc$  and  $t = \frac{X}{c}$  where  $T$  and  $X$  represent the new time and space variables, one finds

$$\beta = -\frac{\beta_4 c^2}{8 \hbar^2 \beta_2}, \quad (11)$$

$$\beta_0 = -\frac{\beta_4 c^5}{8 G \hbar \beta_2}. \quad (12)$$

Since the parameters  $\beta$  and  $\beta_0$  are positive definite, we have the constraint  $\beta_2 \beta_4 < 0$ . Typical values for the parameters  $\beta$  and  $\beta_0$  can be obtained by considering an optical fiber with dispersion coefficients  $\beta_2 = 0.49 \text{ ps}^2/\text{km}$  and  $\beta_4 = -1.1 \times 10^{-7} \text{ ps}^4/\text{m}$  [15] which gives

$$\beta \simeq 10^{56} \frac{\text{s}^2}{\text{kg}^2 \text{m}^2}, \quad \beta_0 \simeq 10^{57}.$$

As we shall see later, G-UP predicts the existence of a maximal localization length corresponding (in the temporal case) to a minimum time uncertainty  $\Delta T_{\min} = \hbar \sqrt{\beta}/c$ . For  $\beta \simeq 1 \frac{\text{s}^2}{\text{kg}^2 \text{m}^2}$ , one finds  $\Delta T_{\min} \simeq \hbar/c \simeq 10^{-42} \text{ s}$  which gives the maximal temporal resolution. While for our analogue, for the value of  $\beta$  given above we find,  $\Delta T_{\min} \simeq 10^{-15} \text{ s}$ . This means that maximally localized states of quantum gravity correspond to optical pulses of duration on the order of femtoseconds and demonstrates that laboratory emulations of the physics at the Planck scale are indeed accessible.

## 3. Optical G-UP: a unified framework

In modern quantum gravity theories, G-UP is typically considered as a postulate. Our goal here is to show that the G-UP formalism is also relevant for spatial and temporal optical wave propagation. We hence follow a different strategy and derive the

generalized uncertainty relation starting from the governing dynamical evolution equation. Thus, the starting point is the normalized higher order nonparaxial optical wave equation

$$i \partial_z \psi + \frac{1}{2} \partial_x^2 \psi - \frac{\varepsilon}{8} \partial_x^4 \psi = 0, \quad (13)$$

where  $\psi$  is the envelop wave-function proportional to the electric field,  $z$  is the propagation direction,  $x$  represents either the spatial or temporal variable and  $\varepsilon > 0$  is a dimensionless parameter that measures the deviation from the paraxial theory. In the spatial case  $\varepsilon = 1/k_0 Z_d$ , where  $Z_d$  is the diffraction length. On the other hand, for temporal pulses  $\varepsilon = -\beta_4/(3\beta_2 T_0^2)$ , with  $T_0$  being the initial temporal pulse period.

Throughout the rest of the paper the forward Fourier transform is defined by

$$F(f) = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}, \quad (14)$$

with the inverse given by

$$f(x) = F^{-1}(\tilde{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx}. \quad (15)$$

The first step in obtaining the generalized optical uncertainty relation is to define the generalized ‘‘momentum’’  $K$ . Taking the Fourier transform of Eq. (13) we obtain

$$i \partial_z \tilde{\psi} - \left( \frac{k^2}{2} + \varepsilon \frac{k^4}{8} \right) \tilde{\psi} = 0, \quad (16)$$

where  $\tilde{\psi}$  is the Fourier transform of  $\psi$ . Defining  $K^2 \equiv k^2 + \varepsilon k^4/4$ , Eq. (16) then takes the equivalent form

$$i \partial_z \tilde{\psi} - \frac{K^2}{2} \tilde{\psi} = 0, \quad (17)$$

with generalized momentum  $K$  approximately given by  $K \approx k + \frac{\varepsilon}{8} k^3$  (for  $\varepsilon \ll 1$  or for limited band-width). In this regard, the inverted dispersion relation reads

$$k \simeq K - \frac{\varepsilon}{8} K^3. \quad (18)$$

We remark that the  $\varepsilon$  expansion has limited the values of the transverse wavevector  $k$ . This in turn would set certain limits on the accessible values for the generalized momentum  $K$  as well. In the new generalized  $K$ -space the scalar product takes the form

$$\langle \tilde{\psi}(K) | \tilde{\phi}(K) \rangle = \int_{-\infty}^{\infty} \frac{\tilde{\psi}^*(K) \tilde{\phi}(K)}{1 + \varepsilon K^2/8} dK. \quad (19)$$

In order to derive the desired uncertainty principle, we first recall the Heisenberg–Robertson inequality [16]: for two operators  $\hat{A}$  and  $\hat{B}$  with uncertainty  $\Delta A$  and  $\Delta B$  we have

$$\Delta A \Delta B \geq \frac{1}{2} \left| \left[ \hat{A}, \hat{B} \right] \right|. \quad (20)$$

The generalized uncertainty principle can be obtained using the following commutation rule:

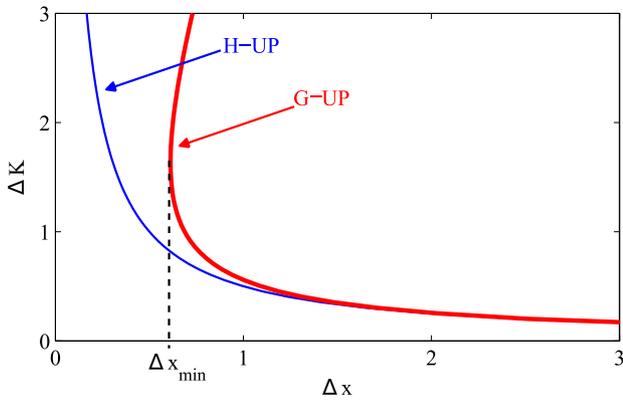
$$\left[ \hat{x}, f(\hat{k}) \right] = i \frac{\partial f(\hat{k})}{\partial k}. \quad (21)$$

With this at hand, we have the following result

$$\left[ \hat{x}, \hat{K}(\hat{k}) \right] = i \left( 1 + 3\varepsilon \hat{k}^2/8 \right). \quad (22)$$

Substituting the expression for  $k$  (see Eq. (18)) in Eq. (22) and keeping terms up to order  $\varepsilon$  we find

$$\left[ \hat{x}, \hat{K}(\hat{k}) \right] = i \left( 1 + 3\varepsilon \hat{K}^2/8 \right), \quad (23)$$



**Fig. 1.** Heisenberg uncertainty principle (thin line) and its generalization (thick line). Note that, in the generalized case, increasing the value of  $\Delta K$  does not imply a reduction in  $\Delta x$  and a minimum  $\Delta x_{\min}$  exist. We used a large value of  $\varepsilon$  ( $\varepsilon = 1$ ) to highlight the differences between the two lines.

from which we obtain

$$\Delta x \Delta K \geq \frac{1}{2} \left( 1 + \frac{3}{8} \varepsilon \langle \hat{K}^2 \rangle \right). \quad (24)$$

If one assumes that  $\langle \hat{K} \rangle = 0$  then the inequality (24) reduces to

$$\Delta x \Delta K \geq \frac{1}{2} \left( 1 + \frac{3}{8} \varepsilon (\Delta K)^2 \right). \quad (25)$$

This is the generalized optical uncertainty principle associated with Eq. (13) given in dimensionless form. In Fig. 1 we show the dependence of  $\Delta K$  on  $\Delta x$ . An important consequence of Eq. (25) is the existence of a minimal position uncertainty

$$\Delta x_{\min} = \sqrt{\frac{3\varepsilon}{8}}. \quad (26)$$

We remark that this is valid for  $\varepsilon \ll 1$  or for a limited bandwidth. This theory predicts maximally localized states, which are the ones that satisfy strictly the generalized uncertainty principle and hence have a width equal to  $\Delta x_{\min}$ . On the other hand we can see that this theory agrees with the Heisenberg uncertainty principle. Indeed, when  $\varepsilon = 0$  and for a fixed  $\Delta K$ , it is possible to focus a given beam with a spatial width satisfying  $\Delta x = \frac{1}{2} \frac{1}{\Delta K}$  in which case  $\Delta x_{\min} \rightarrow 0$  as  $\Delta K \rightarrow \infty$ .

### 3.1. Gaussian wave-packets and minimal uncertainty

In this section we apply the results obtained so far for a chirped Gaussian beam [3] by calculating its uncertainty relation  $\Delta x \Delta K$ . We assume a wave-function in the form

$$\psi(x) = \frac{1}{\sqrt{\pi x_0}} e^{-\frac{x^2}{2x_0^2}(1+iC)}, \quad (27)$$

where  $C$  is a chirp parameter (tilt in the spatial case) and  $x_0$  measures the beam waist. Its corresponding form in momentum space is given by

$$\tilde{\psi}(k) = \frac{\sqrt{x_0}}{\sqrt[4]{\pi}} \frac{1}{1+iC} e^{-\frac{k^2 x_0^2}{2(1+iC)}}. \quad (28)$$

Straightforward calculations show

$$(\Delta x)^2 = \int x^2 |\psi|^2 dx = \frac{1}{2} x_0^2, \quad (29)$$

and

$$(\Delta k)^2 = \int k^2 |\tilde{\psi}|^2 dk = \frac{1}{2} \frac{1+C^2}{x_0^2}, \quad (30)$$

**Table 2**  
Minimal  $\Delta x$  for standard and generalized uncertainty principle, with a chirp parameter  $C$ .

	H-UP	G-UP
$\Delta x$	0	$\Delta x_{\min}$
$\Delta x$ for $C \rightarrow \infty$	0	$\infty$

which imply [3]

$$\Delta x \Delta k = \frac{1}{2} \sqrt{1+C^2}. \quad (31)$$

Scrutinizing expression (31) shows that in the absence of a fourth order dispersion/diffraction no minimal scale exists. Indeed, when  $\Delta k$  gets larger the uncertainty in position can be made arbitrarily small. If one solves Eq. (13) with  $\varepsilon = 0$  and using the Gaussian wavepacket given in (27) as an input, then one finds the minimal beam waist to be

$$\Delta x_{\min} = \frac{x_0}{\sqrt{2} \sqrt{1+C^2}}. \quad (32)$$

Importantly, the uncertainty in position, i.e.,  $\Delta x_{\min}$  tends to zero as the chirp  $C \rightarrow \infty$ . Of course, the situation is different when adding nonparaxial effects. With the definition of the generalized momentum  $K = k + \frac{\varepsilon}{8} k^3$ , we next compute the uncertainty  $(\Delta K)^2$ . To leading order in  $\varepsilon$  we have

$$\begin{aligned} (\Delta K)^2 &= \int K^2 |\tilde{\psi}(K)|^2 \frac{dK}{1 + \frac{3}{8} \varepsilon K^2} \\ &\approx \int dk \left( k + \frac{\varepsilon}{8} k^3 \right)^2 |\tilde{\psi}(k)|^2 \\ &\approx \frac{1}{2} \frac{(1+C^2)}{x_0^2} \left( 1 + \frac{3\varepsilon(1+C^2)}{8x_0^2} \right). \end{aligned} \quad (33)$$

With this at hand, the generalized uncertainty principle reads

$$\begin{aligned} \Delta x \Delta K &= \frac{1}{2} \sqrt{1+C^2} \sqrt{1 + \frac{3\varepsilon(1+C^2)}{8x_0^2}} \\ &\simeq \frac{1}{2} \sqrt{1+C^2} \left( 1 + \frac{3\varepsilon(\Delta K)^2}{8} \right), \end{aligned} \quad (34)$$

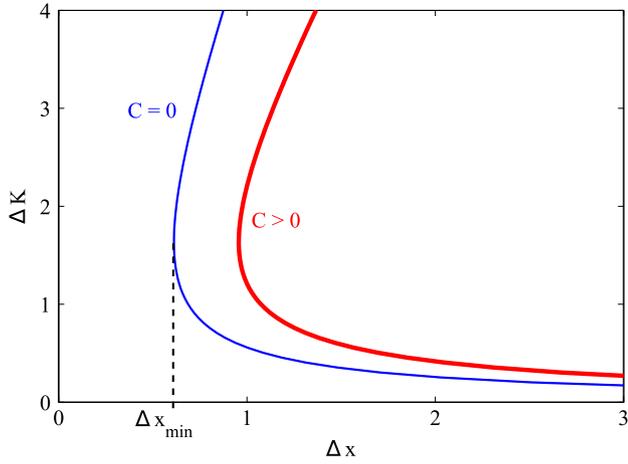
which is valid to the leading order in  $\varepsilon$ . Fig. 2 depicts the dependence of the uncertainty in momentum to that in physical space in the presence of a chirp and fourth order dispersion/diffraction—note the existence of a minimal scale. Expression (34) agrees with the general uncertainty principle (25) for  $\varepsilon \neq 0$ . Moreover if  $\varepsilon = 0$  and the chirp  $C$  tends to zero, we recover the standard Heisenberg relation. From Eq. (34), it follows that the minimal value of  $\Delta x$  is

$$\Delta x_{\min}^\varepsilon = \sqrt{\frac{3}{8} \varepsilon (1+C^2) + o(\varepsilon^2)}. \quad (35)$$

Note that when  $C \rightarrow \infty$ ,  $\Delta x_{\min} \rightarrow \infty$  which means that there is a minimal value of  $\Delta x_{\min}$  which is the one found previously in Eq. (26) [see Table 2].

### 3.2. The generalized position operator

In standard quantum mechanics, eigenstates of the position operator  $\hat{x}$ , corresponding to ideally localized wave-functions with  $\Delta x = 0$ , form a basis of the Hilbert space. In the G-UP literature, states with  $\Delta x = 0$  are not physically acceptable as the position operator is not self-adjoint. In this framework one considers the *maximally localized states*, that satisfy Eq. (26), i.e.,  $\Delta x = \Delta x_{\min}$ , as *quasi-position eigenstates*. In our analogy, these states correspond to the mostly localized beams (within the adopted first order



**Fig. 2.** Generalized uncertainty principle in the presence of a chirp  $C$ . For visualization purposes we take  $\varepsilon = 1$ .

nonparaxial approximation) or to the shortest light pulses one can achieve in the presence of second and fourth order dispersion.

Our aim here is to derive the maximal localized states satisfying Eq. (13). Our approach closely follows that of [17]. We start from the eigenstates of the generalized momentum operator  $\hat{K}$

$$\hat{K}\tilde{\psi}(K) = K\tilde{\psi}(K). \quad (36)$$

The action of the operator  $\hat{x}$  in the  $K$  basis is given by

$$\hat{x}\tilde{\psi}(K) = i\left(1 + \frac{3}{8}\varepsilon K^2\right)\partial_K\tilde{\psi}(K), \quad (37)$$

which gives the correct GUP result. Indeed we find

$$[\hat{x}, \hat{K}]\tilde{\psi}(K) = i\left(1 + \frac{3}{8}\varepsilon K^2\right)\tilde{\psi}(K). \quad (38)$$

The operators  $\hat{x}$  and  $\hat{K}$  are symmetric with respect to Eq. (19), that is

$$\langle\psi|\hat{K}|\phi\rangle = \langle\psi|\left(\hat{K}|\phi\rangle\right), \quad (39)$$

$$\langle\psi|\hat{x}|\phi\rangle = \langle\psi|\left(\hat{x}|\phi\rangle\right). \quad (40)$$

The following completeness and orthogonality relations hold true:

$$1 = \int_{-\infty}^{+\infty} \frac{dK}{1 + \beta K^2} |K\rangle\langle K| \quad (41)$$

$$\langle K|K'\rangle = (1 + \beta K^2)\delta(K - K'). \quad (42)$$

In order to compute the eigenstates of the  $\hat{x}$  operator, we consider the following eigenvalue problem in the  $K$  space

$$\hat{x}\tilde{\psi}(K) = \lambda\tilde{\psi}(K), \quad (43)$$

where after the substitution  $\hat{x} = i(1 + \beta K^2)\partial_K$  becomes

$$i(1 + \beta K^2)\partial_K\tilde{\psi}(K) = \lambda\tilde{\psi}(K). \quad (44)$$

Solving this equation gives the normalized position eigenfunction in the  $K$  representation ( $\lambda \in \mathbb{R}$ )

$$\tilde{\psi}(K) = \sqrt{\frac{\sqrt{\beta}}{\pi}} e^{-\frac{\lambda}{i\sqrt{\beta}} \arctan(K\sqrt{\beta})}. \quad (45)$$

In the case of complex eigenvalue  $\lambda$  ( $\lambda \in \mathbb{C}$ ), the normalized eigenfunction is given by

$$\tilde{\psi}(K) = \sqrt{\frac{\text{Im}(\lambda)}{\sinh\left[\frac{\pi \text{Im}(\lambda)}{\sqrt{\beta}}\right]}} e^{-\frac{\lambda}{i\sqrt{\beta}} \arctan(K\sqrt{\beta})}. \quad (46)$$

We remark that  $\lambda$  can be a complex number since  $\hat{x}$  is not self-adjoint.

#### 4. Evaluation of the maximally localized states

In this section our aim is to construct the maximally localized states (MLS) in the physical space representation. Our approach is similar to that of [17]. Such a state  $\psi_{\xi}^{ML}$  is defined by:

$$\langle\psi_{\xi}^{ML}|\hat{x}|\psi_{\xi}^{ML}\rangle = \xi \quad (47)$$

$$\langle\Delta x\rangle_{\psi_{\xi}^{ML}} = \Delta x_{\min}. \quad (48)$$

Following Heisenberg [18], we start from

$$\left\|\left(x - \langle x \rangle + \frac{\langle [x, K] \rangle}{2(\Delta K)^2} (K - \langle K \rangle)\right) |\psi\rangle\right\| \geq 0, \quad (49)$$

which implies that

$$\Delta x \Delta K \geq \frac{|\langle [x, K] \rangle|}{2}. \quad (50)$$

If a state  $|\psi\rangle$  satisfies  $\Delta x \Delta K = |\langle [x, K] \rangle|/2$ , we have

$$\left(x - \langle x \rangle + \frac{\langle [x, K] \rangle}{2(\Delta K)^2} (K - \langle K \rangle)\right) |\psi\rangle = 0. \quad (51)$$

Therefore Eq. (51) is used to find the MLS. Hereafter, we use the following notation:

$$\langle\hat{x}\rangle = \xi \quad (52)$$

$$\langle\hat{K}\rangle = K_1 \quad (53)$$

$$\langle\hat{K}^2\rangle = K_2 \quad (54)$$

$$[\hat{x}, \hat{K}] = i(1 + \beta K^2). \quad (55)$$

From Eq. (51) we have in the  $K$  space

$$i\partial_K\tilde{\psi} = \frac{-i}{1 + \beta K^2} \left[ \xi - \frac{i(1 + \beta K_2^2)}{2(\Delta K)^2} (K - K_1) \right] \tilde{\psi}. \quad (56)$$

Using the method of separation of variables, we find

$$\tilde{\psi}(K) = \tilde{\psi}(0) \exp \left[ \left( -\frac{i\xi}{\sqrt{\beta}} + \frac{(1 + \beta K_2^2) K_1}{2(\Delta K)^2 \sqrt{\beta}} \right) \arctg(\sqrt{\beta}K) \right] \times \left[ 1 + \beta K^2 \right]^{-\frac{1 + \beta K_2^2}{2(\Delta K)^2} \frac{1}{2\beta}}. \quad (57)$$

For  $\langle K \rangle = 0$  and  $\Delta K = \sqrt{K_2} = 1/\sqrt{\beta}$  we obtain

$$\tilde{\psi}^{ML}(K) = \tilde{\psi}(0) \frac{e^{-\frac{i\xi}{\sqrt{\beta}} \arctg(\sqrt{\beta}K)}}{(1 + \beta K^2)^{1/2}} \quad (58)$$

where  $\xi \in \mathbb{R}$ . Imposing the normalization  $\langle\psi^{ML}|\psi^{ML}\rangle = 1$  we have  $\tilde{\psi}(0) = \sqrt{2\sqrt{\beta}/\pi}$ . From Eq. (58), one can verify that

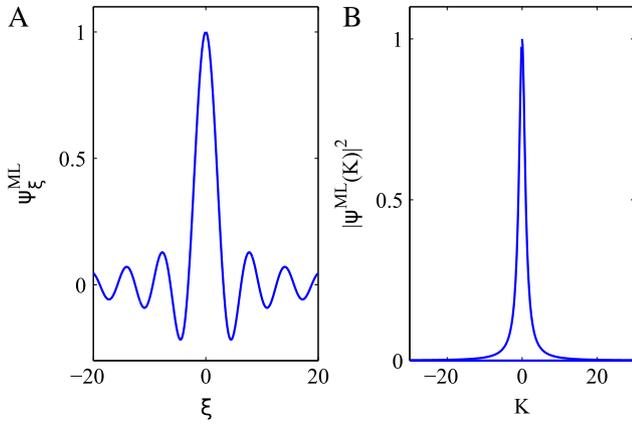
$$\begin{aligned} (\Delta K)^2 &= \frac{1}{\beta} \\ (\Delta x)^2 &= \beta, \end{aligned} \quad (59)$$

for  $\langle x \rangle = \xi = 0$ .

One can also verify that these states have finite energy  $\langle\hat{H}\rangle = K^2 = 1/\beta$ .

These states are not mutually orthogonal, i.e.

$$\langle\psi_{\xi'}^{ML}(K')|\psi_{\xi}^{ML}(K)\rangle \neq \delta_{\xi', \xi}(K' - K). \quad (60)$$



**Fig. 3.** (A) Maximally localized state  $\langle \psi_{\xi} | \psi_0 \rangle$  in the  $\xi$ -space; (B) Square modulus of the generalized Fourier transform of the maximally localized state.

Indeed

$$\begin{aligned} \langle \psi_{\xi'}^{ML} | \psi_{\xi}^{ML} \rangle &= \int \frac{dK}{(1 + \beta K^2)^2} \frac{2\sqrt{\beta}}{\pi} e^{-i(\xi - \xi') \frac{\arctg(\sqrt{\beta}K)}{\sqrt{\beta}}} = \\ &= \frac{1}{\pi} \left[ \frac{\xi - \xi'}{\sqrt{\beta}} - \left( \frac{\xi - \xi'}{\sqrt{\beta}} \right)^3 \right]^{-1} \sin \left( \frac{\xi - \xi'}{\sqrt{\beta}} \pi \right), \end{aligned} \quad (61)$$

so they do not furnish a classical basis as in ordinary quantum mechanics. However they can be used as a representation for wave-functions. Projecting a generic state  $|\phi\rangle$  on  $|\phi^{ML}\rangle$  we have:

$$\begin{aligned} \phi(\xi) &= \langle \psi_{\xi}^{ML} | \phi \rangle \\ &= \int_{-\infty}^{+\infty} dK \frac{2\sqrt{\beta}}{\pi(1 + \beta K^2)^{3/2}} e^{i\xi \frac{\arctg(\sqrt{\beta}K)}{\sqrt{\beta}}} \tilde{\phi}(K). \end{aligned} \quad (62)$$

In standard quantum mechanics this would correspond to the usual Fourier transform. Notably, this generalized Fourier transform is also invertible, as follows

$$\tilde{\phi}(K) = \int_{-\infty}^{+\infty} d\xi \frac{1}{\sqrt{8\pi\sqrt{\beta}}} (1 + \beta K^2)^{1/2} e^{-i\xi \frac{\arctg(\sqrt{\beta}K)}{\sqrt{\beta}}} \phi(\xi). \quad (63)$$

In Fig. 3 we show the characteristic profile of a maximally localized state defined by  $\langle \psi_{\xi} | \psi_0 \rangle$  and its generalized Fourier transform. We remark the presence of the typical oscillations in the maximally localized field.

It is possible to extend the theoretical model to 2D or 3D [17]. Indeed, the minimal uncertainty length theory we considered [17] can be extended to higher dimensions. The issue is that the analogy with photonics is limited to certain conditions. When focalization is pushed over, one cannot limit the analysis to the first order and a more general approach is required. In that case, it can be demonstrated that it is not always possible to reach maximal localization [19]. This might be specifically relevant in case of the catastrophic beam collapse [20] that not only involves higher order diffraction terms but also vectorial effects.

## 5. Generalized uncertainty principle and nonlinearity

In this section we show the way nonlinearity triggers the generation of maximally localized states. For that purpose, we consider nonlinear Schrödinger equation (NLS) with nonlocal nonlinearity and higher order diffraction [21–24]:

$$\begin{aligned} i \frac{\partial \psi}{\partial z} &= -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \frac{\varepsilon}{8} \frac{\partial^4 \psi}{\partial x^4} + \\ &- g \psi \int_{-\infty}^{+\infty} G(x - x') |\psi(x')|^2 dx', \end{aligned} \quad (64)$$

where  $g > 0$  measures the strength of the nonlinearity;  $\varepsilon > 0$  is the higher order diffraction coefficient and  $G(x)$  is a kernel given by

$$G(x) = \frac{e^{-|x|/\sigma}}{2\sigma}, \quad (65)$$

where  $\sigma > 0$  is a constant that characterizes the degree of nonlocality. Bound states for Eq. (64) are sought of in the form  $\psi(x, z) = \phi(x) \exp(i\mu z)$  with  $\phi$  satisfying the boundary value problem

$$\begin{aligned} -\mu \phi &= -\frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\varepsilon}{8} \frac{\partial^4 \phi}{\partial x^4} + \\ &- g \phi \int_{-\infty}^{+\infty} G(x - x') |\phi(x')|^2 dx' \end{aligned} \quad (66)$$

with  $\mu > 0$  being the soliton eigenvalue. Our aim next is to understand how the localization length of the bound states depends on the nonlinearity strength.

### 5.1. Maximally localized nonlinear modes

Solutions to Eq. (66), in the form of localized nonlinear waves, can be numerically computed using the spectral renormalization method [25]. To do so we define the renormalized complex wave function

$$\phi(x) = Ru(x), \quad (67)$$

where, in general,  $R$  is a complex scalar, different from zero. Substituting (67) into (66) gives the expression for the renormalization factor

$$|R(u)|^2 = \frac{\mu N(u) + E_k(u)}{E_{\text{non}}(u)}, \quad (68)$$

where we define the “kinetic”, interaction energy and the power respectively:

$$E_k(u) \equiv \frac{1}{2} \int |u_x|^2 dx + \frac{\varepsilon}{8} \int |u_{xxx}|^2 dx, \quad (69)$$

$$E_{\text{non}}(u) \equiv g \iint G(x - x') |u(x)|^2 |u(x')|^2 dx' dx \quad (70)$$

$$N(u) \equiv \int |u|^2 dx. \quad (71)$$

Using the one-dimensional Fourier transform defined in Eq. (14), we obtain

$$\hat{u} = \frac{g|R|^2 F(u) * [F(G)F(|u|^2)]}{\mu + k^2/2 + \varepsilon k^4/8}, \quad (72)$$

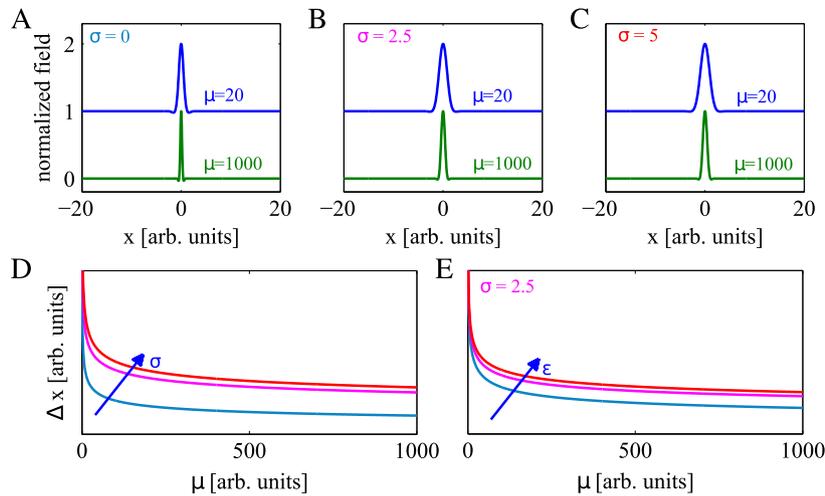
where  $*$  denotes the standard Fourier convolution operation. Eq. (72) is a fixed point equation for  $\hat{u}$  which can be solved by a direct fixed point iteration

$$\hat{u}_{n+1} = Q(\hat{u}_n, \mu_n, |R_n|^2), \quad (73)$$

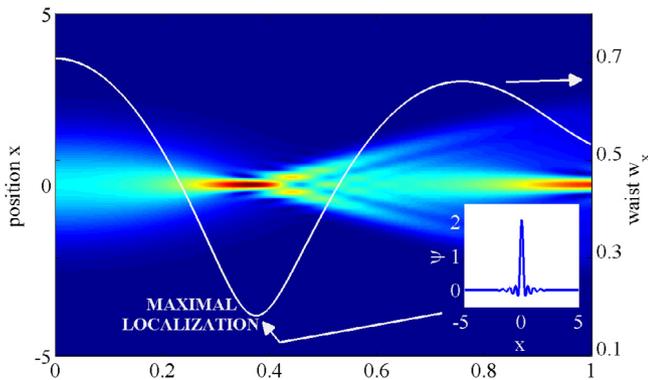
where  $|R_n|^2 \equiv |R(u_n)|^2$  and

$$Q(\hat{u}, \mu, |R|^2) = \frac{g|R|^2 F(u) * [F(G)F(|u|^2)]}{\mu + k^2/2 + \varepsilon k^4/8}.$$

In Fig. 4 we show typically bound states obtained from the spectral renormalization method. At fixed nonlinearity  $g$ , we computed the soliton width for various values of eigenvalue  $\mu$ . This is equivalent to varying the soliton power. We observe that at high  $\mu$  the wave profile develops lateral lobes (bottom curve in panels A, B and C of Fig. 4) as expected for the maximally localized state (see Fig. 3(A)). These lobes become smoother as the degree of nonlocality  $\sigma$  gets larger. In panels D and E of Fig. 4 we show the dependence of the soliton width on power. As one can see  $\Delta x$  increases for higher values of  $\sigma$ . The same result is obtained by varying the degree of nonparaxiality  $\varepsilon$ . It is worth to note that for increasing  $\mu$  the soliton width saturates and achieves its maximal localization length.



**Fig. 4.** (A) Normalized fields  $u$  after Eqs. (66) and (67), for high and low values of the eigenvalues  $\mu$  ( $\mu = 20$  or  $1000$ ) in the case of local nonlinearity. The curve at low  $\mu$  has been shifted on the vertical axes to allow a clearer view of the lobes. (B) and (C) as in (A) for degree of nonlocality  $\sigma = 2.5, 5$  respectively. (D) Behavior of the width  $\Delta x$  as a function of the eigenvalue  $\mu$  for different nonlocality  $\sigma$  ( $\sigma = 0, 2.5, 5$ ); (E) as in (D) but at different values of the degree of nonparaxiality ( $\epsilon = 0, 0.5, 1$ ) at fixed nonlocality  $\sigma = 2.5$ . We used large values for  $\epsilon$  to emphasize the differences among the lines.



**Fig. 5.** Simulation of a Gaussian beam evolving according to Eq. (66), for  $g = 1$ ,  $\epsilon = 10^{-5}$ ,  $\mu = 11 \times 10^4$  and  $\sigma = 5$ . The superimposed white line shows the waist  $w_x$  versus the propagation direction  $z$ . The inset shows the field profile at the point of maximal localization ( $z \simeq 0.4$ ).

## 5.2. Excitation of maximally localized states

In order to provide a further evidence that nonlinearity forces the system towards maximal localization, we numerically solve Eq. (64) with kernel (65). The initial beam profile is a Gaussian beam [see Eq. (27)]. Fig. 5 shows that the beam focuses upon propagation and its waist  $w_x$  presents a minimum (maximal localization) during propagation. As the inset shows, the field at the maximal localization displays the characteristic lateral lobes, with a remarkable resemblance to Fig. 3(A).

Albeit, these results confirm the onset of maximal localization, we remark that when the beam waist is comparable with the minimal length the first order perturbation theory used in Eq. (13) loses validity. This calls for more advanced theoretical methods that will be reported in future works.

## 6. Conclusions

We have reported on the implementation of the quantum gravity generalized uncertainty principle in the nonlinear Schrödinger equation and provided an analogue to study quantum gravity effects in optical wave propagation. We considered the simplest form of the theory based on a generalized linear Schrödinger equation with higher order dispersion/diffraction. This equation describes the propagation of ultra-short pulses in fibers or one-dimensional

sub-paraxial focused beams. We discussed the way a generalized uncertainty principle enters in the description of possible states. We analyzed the resulting maximally localized states and have shown the way they can be excited in nonlinear propagation. Our goal was to demonstrate that ideas from quantum gravity have relevance in optics and photonics including the nonlinear regime. This analysis might be extended in several directions such as retaining higher order dispersion and calculating the shortest pulse that can propagate in a fiber at any dispersion order. Another possibility might be designing spatially modulated beams in order to ultra-focus beyond the limits imposed by standard numerical aperture.

Developments also include novel classes of nonlinear waves in the spatio-temporal domain. Furthermore our results show that photonics can be an important framework to realize analogues or models of quantum gravity theories [26–29].

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