Theory of coupled optical $\mathcal{PT}$-symmetric structures

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Received June 6, 2007; accepted July 12, 2007;
posted July 31, 2007 (Doc. ID 83814); published August 31, 2007

Starting from Lagrangian principles we develop a formalism suitable for describing coupled optical parity-time symmetric systems. © 2007 Optical Society of America

OCIS codes: 130.2790, 230.7370.

Physical systems exhibiting parity-time ($\mathcal{PT}$) symmetry have been the subject of intense investigation in the past few years [1–4]. It has been shown in a series of studies that $\mathcal{PT}$-symmetric Hamiltonians can have a real eigenvalue spectrum—a surprising result given that in general these Hamiltonians are non-Hermitian [1]. Another intriguing aspect associated with this family of pseudo-Hermitian configurations is the possibility of an abrupt phase transition (from a real to a complex spectrum) because of a spontaneous breakdown of $\mathcal{PT}$ symmetry. Following the work of Bender *et al.* [2], an operator $\hat{A}$ is $\mathcal{PT}$ symmetric if it shares a common set of eigenvectors with the $\hat{P}\hat{T}$ operator, in which case $[\hat{A},\hat{P}\hat{T}]=0$. Here the parity operator $\hat{P}$ is defined as $\hat{P}:\hat{x}\to-\hat{x}$, $\hat{p}\to-\hat{p}$ while the time reversal operator leads to $\hat{T}:\hat{p}\to-\hat{p}$, $i\to-i$, where $\hat{p},\hat{x}$ are the momentum and the position operators, respectively. From this latter requirement one can show that the potentials associated with these pseudo-Hermitian Schrödinger problems must satisfy the condition $V(x)=V^*(-x)$ [2].

In optics, such complex $\mathcal{PT}$-symmetric structures can be realized within the context of the paraxial theory of diffraction by involving *symmetric index* guiding and an *antisymmetric gain/loss* profile, that is, $n(x)=n^*(-x)$. In other words, the index and gain guiding in such configurations must be judiciously realized. In these systems, the electric field envelope obeys a normalized complex Schrödinger equation, e.g.,

$$i\frac{\partial\phi(\eta,\xi)}{\partial\xi}+\frac{\partial^2\phi(\eta,\xi)}{\partial\eta^2}+V(\eta)\phi(\eta,\xi)=0, \tag{1}$$

where $\xi=z/(2kx_0^2)$ is a scaled propagation distance, $\eta=x/x_0$ is a dimensionless transverse coordinate, and x_0 is an arbitrary spatial scale. Here, $k=2\pi n_0/\lambda_0$, n_0 is the background refractive index, and $V(\eta)=(2k^2x_0^2/n_0)n(\eta)$ represents the normalized complex index distribution that satisfies the $\mathcal{PT}$ condition. Note that in this physical model, the propagation distance ξ plays the role of time in quantum mechanics. Given the fact that $\mathcal{PT}$ arrangements may provide an additional degree of freedom in synthesizing novel optical structures and materials, it would be of great interest to study their optical behavior and characteristics. One fundamental aspect associated with $\mathcal{PT}$ components has to do with their coupled-mode interactions.

In this Letter we formulate a coupled-mode theory (CMT) appropriate for $\mathcal{PT}$-symmetric optical elements, i.e., when each individual element as well as the entire system respects $\mathcal{PT}$ symmetry. This is done through a Lagrangian treatment of the problem and by employing the particular inner product of these systems. As we will see, this new formulation is necessary since the conventional CMT fails in this regime. Pertinent examples are provided to demonstrate the validity of our results.

We begin our analysis by considering the action of the $\hat{P}\hat{T}$ operator on Eq. (1), which yields

$$-i\frac{\partial\phi^*(-\eta,\xi)}{\partial\xi}+\frac{\partial^2\phi^*(-\eta,\xi)}{\partial\eta^2}+V(\eta)\phi^*(-\eta,\xi)=0. \tag{2}$$

Note that in Eq. (2) $V(\eta)$ remains invariant as a result of the assumed $\mathcal{PT}$ symmetry. From Eqs. (1) and (2) one can readily obtain the first conservation law of the system; i.e., $Q=\int_{-\infty}^{\infty}\phi(\eta,\xi)\phi^*(-\eta,\xi)\mathrm{d}\eta$ is a constant of motion independent of distance ξ [5]. Note that, in contrast to conventional optical systems, this latter conserved quantity does not represent the actual power. To obtain the equations of motion describing the coupling interaction between $\mathcal{PT}$ elements, we employ the Lagrangian density associated with Eq. (1), which is given by

$$L=\frac{i}{2}[\phi(\eta)\phi_\xi^*(-\eta)-\phi_\xi(\eta)\phi^*(-\eta)]+\phi_\eta(\eta)\phi_\eta^*(-\eta)-V(\eta)\phi(\eta)\phi^*(-\eta). \tag{3}$$

Note that variation $\delta L/\delta\phi(\eta,\xi)=\partial_\xi(\partial L/\partial\phi_\xi)+\partial_\eta(\partial L/\partial\phi_\eta)-\partial L/\partial\phi=0$ leads to Eq. (2) while $\delta L/\delta\phi^*(-\eta,\xi)=0$ gives Eq. (1). In addition, the Hamiltonian invariant of this system is given by $\int_{-\infty}^{\infty}H\mathrm{d}\eta$, where $H=V(\eta)\phi(\eta)\phi^*(-\eta)-\phi_\eta(\eta)\phi_\eta^*(-\eta)$. We would like to emphasize that the two conserved quantities Q and the Hamiltonian H are not related to the actual optical power (defined as $P=\int_{-\infty}^{\infty}\phi(\eta,\xi)\phi^*(\eta,\xi)\mathrm{d}\eta$) or the energy of the system. In fact even below phase transition, where the $\mathcal{PT}$ system has a real spectrum, the power P is not necessarily conserved.

Next, let us consider two identical coupled $\mathcal{PT}$ waveguide elements as shown schematically in Fig.

1(a). Note that not only each element is locally $\mathcal{PT}$, but also the entire structure satifies $\mathcal{PT}$ symmetry with respect to the geometric axis M at $\eta=0$. Here, for simplicity and without any loss of generality, we will consider a 1D configuration. In addition we assume that the system is used below the phase transition point (in the real spectrum domain). We proceed by further assuming that the solution of this coupled $\mathcal{PT}$ arrangement can be expressed as a superposition of the local modes of the individual elements, i.e.,

$$\phi(\eta,\xi)=[a(\xi)u_1(\eta)+b(\xi)u_2(\eta)]\exp(i\mu\xi), \qquad (4)$$

where $u_{1,2}(\eta)$ represents the local eigenfunctions of these two waveguides and μ stands for their corresponding real propagation eigenvalue because of $\mathcal{PT}$ symmetry. By substituting Eq. (4) into the Lagrangian density of Eq. (3) and by integrating over η we obtain the reduced Lagrangian of this system $\langle L\rangle=\int_{-\infty}^{\infty}L\mathrm{d}\eta$ as a function of the modal amplitudes $a(\xi),a^*(\xi),b(\xi),b^*(\xi)$, and their respective derivatives, that is, $\dot{a}=\mathrm{d}a/\mathrm{d}\xi$, etc. The two coupled-mode equations can then be obtained by extremizing the reduced Lagrangian with respect to the modal amplitudes. By doing so we find, $i\dot{a}I_{12}+i\dot{b}I_{22}+aJ_{212}+bJ_{122}=0$ and $i\dot{b}I_{21}+i\dot{a}I_{11}+bJ_{121}+aJ_{211}=0$. The first equation was obtained from $\delta\langle L\rangle/\delta b^*=0$ and the second from $\delta\langle L\rangle/\delta a^*=0$. In these latter equations, $I_{km}=\int_{-\infty}^{\infty}u_k(\eta)u_m^*(-\eta)\mathrm{d}\eta$ and $J_{jkm}=\int_{-\infty}^{\infty}V_j(\eta)u_k(\eta)u_m^*(-\eta)\mathrm{d}\eta$. Because of the reflection $(-\eta)$ used in these inner products (overlap integrals) and the localization of the eigenfunctions u_1 and u_2 around their respective potentials [Fig. 1(b)], one finds that $I_{12}\gg I_{22}$ and $I_{21}\gg I_{11}$. As a result, the coupled-mode equations describing this $\mathcal{PT}$ symmetric system are given by

$$i\frac{da}{d\xi}+\Delta_a a+\kappa b=0 \qquad (5a)$$

$$i\frac{db}{d\xi}+\Delta_b b+\kappa a=0, \qquad (5b)$$

where κ is the coupling coefficient ($\kappa=J_{122}/I_{12}=J_{211}/I_{21}$) and $\Delta_{a,b}$ represent shifts in the propagation constants ($\Delta_a=J_{212}/I_{12}$ and $\Delta_b=J_{121}/I_{21}$) as a result of the coupling interaction.

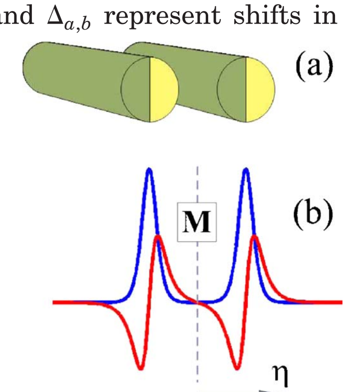

Fig. 1. (Color online) $\mathcal{PT}$-coupled waveguide system: (a) waveguide configuration (green represents gain region while yellow stands for loss region) and (b) refractive index (blue) and gain/loss profile (red). M stands for the geometric symmetry axis.

We next show that the coupling constant in such an arrangement is real. From the assumed $\mathcal{PT}$ symmetry shown in Fig. 1(b) it is evident that $u_1(\eta)=u_2^*(-\eta)$. Expressing u_1 in terms of its real and imaginary parts, $u_1=u_{1R}+iu_{1I}$, we get $I_{12}=\int_{-\infty}^{\infty}u_1^2(\eta)\mathrm{d}\eta=\int_{-\infty}^{\infty}[u_{1R}^2(\eta)-u_{1I}^2(\eta)+2iu_{1R}(\eta)u_{1I}(\eta)]\mathrm{d}\eta$. Since for a $\mathcal{PT}$-symmetric potential $u_{1R}(\eta)$ is an even/odd function while $u_{1I}(\eta)$ is an odd/even function with respect to its local center, it turns out that I_{12} is a real quantity and so is I_{21}. We will next prove that both J_{122} and J_{211} are real. To do so, we consider the evolution equation associated with the first potential in isolation, i.e., Eq. (1) with $V(\eta)=V_1(\eta)$. By substituting the stationary solution $\varphi(\eta,\xi)=u_1(\eta)\exp(i\mu\xi)$ in this equation we get $-\mu u_1(\eta)+u_{1\eta\eta}(\eta)+V_1(\eta)u_1(\eta)=0$. Multiplying this latter expression by $u_2(\eta)$ and integrating by parts, we get $\int V_1(\eta)u_1(\eta)u_2(\eta)\mathrm{d}\eta=\mu\int u_1(\eta)u_2(\eta)\mathrm{d}\eta+\int u_{1\eta}(\eta)u_{2\eta}(\eta)\mathrm{d}\eta$. Noting that $u_1(\eta)=u_2^*(-\eta)$ one obtains $J_{122}=\int V_1(\eta)u_2(\eta)u_2^*(-\eta)d\eta=\int V_1(\eta)u_2(\eta)u_1(\eta)d\eta$, in other words $J_{122}=\mu\int u_1(\eta)u_2(\eta)d\eta+\int u_{1\eta}(\eta)u_{2\eta}(\eta)d\eta$. Using $\mathcal{PT}$ symmetry it is readily shown that $(\int u_1(\eta)u_2(\eta)d\eta)^*=\int u_1(\eta)u_2(\eta)d\eta$ and $(\int u_{1\eta}(\eta)u_{2\eta}d\eta)^*=\int u_{1\eta}(\eta)u_{2\eta}d\eta$, i.e., they are both real quantities and so is J_{122}. The reality of J_{211} is also guaranteed because of symmetry, e.g., $J_{211}=J_{122}$. Thus the coupling constant $\kappa=J_{122}/I_{12}=J_{211}/I_{21}$ happens to be real. Finally, by considering the $\mathcal{PT}$ symmetry of the coupled structure, it is straightforward to show that J_{212} and J_{121} are complex conjugates of each other and so are the perturbations introduced in the propagation constant of each waveguide, e.g., $\Delta_a=\Delta_b^*$.

In retrospect, we could have arrived at this same formalism by projecting the evolution equations on the $\mathcal{PT}$-symmetric base functions $u_{2,1}^*(-\eta)$ as opposed to $u_{1,2}^*(\eta)$ used in conventional CMT [6,7]. We also note that had we used the standard coupled-mode equations it would have instead resulted into a complex coupling constant and real propagation constant shifts $\Delta_{a,b}$.

We will now illustrate our results using relevant examples. Figure 2 (inset) shows the evolution of an input optical beam in a $\mathcal{PT}$-symmetric coupler when each potential in isolation has the form $V(\eta)=A^2\,\mathrm{sech}^2(\eta\pm D/2)+iB\,\mathrm{sech}(\eta\pm D/2)\tanh(\eta\pm D/2)$, where $A=\sqrt{2+B^2/9}$ and D is the separation between the two potentials. The choice for these particular potentials is motivated by their analytical solutions, which in this case are given by $\mathrm{sech}(\eta\pm D/2)\exp\{i(B/3)\tan^{-1}[\sinh(\eta\pm D/2)]\}$. Using the formalism developed above, we have computed the normalized coupling length for various separations D. These results are in excellent agreement with numerical results obtained from supermode analysis and beam propagation methods as shown in Fig. 2. We note that the coupled evolution in this example is affected by the fact that Δ_a and Δ_b are complex conjugates of each other (effectively one arm exhibits gain and the other loss). Figure 3 on the other

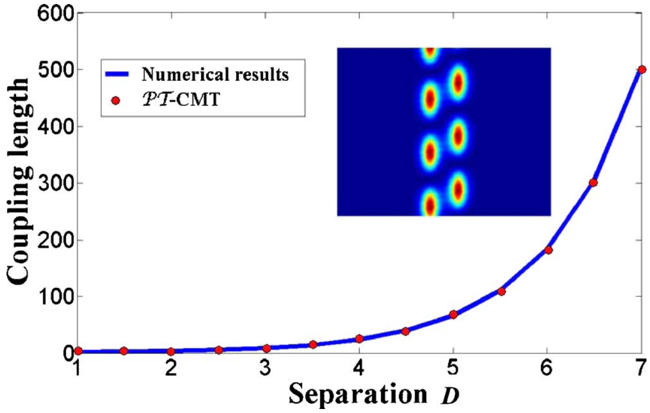

Fig. 2. (Color online) Normalized coupling length calculated from supermode analysis (solid curve) compared with that obtained from the $\mathcal{PT}$-CMT (dots) as a function of waveguide separation D. Inset shows a simulation of beam propagation when the separation between the two waveguides is $D=4$.

hand depicts the evolution of a single channel excitation in an array made of $\mathcal{PT}$ potentials (below phase transition), used in the previous example. In this configuration, the phase-shift of the diffracted beams at the left and right channels (symmetrically located around the excitation site) is compared and found to be zero. This in excellent agreement with the predictions of the CMT derived here that suggests a real coupling constant. Because of this arrangement, in

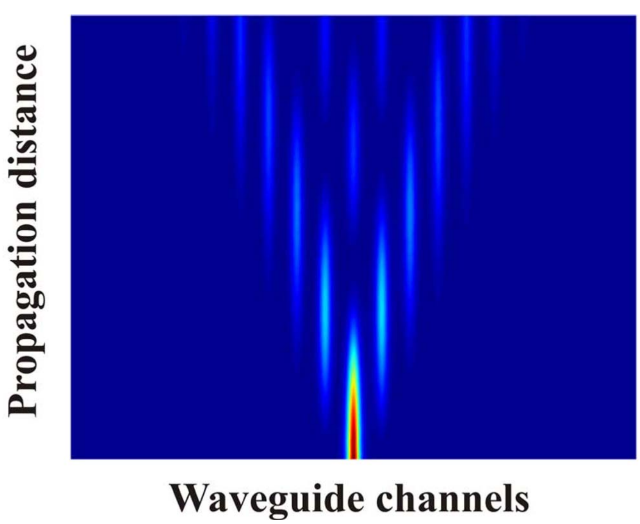

Fig. 3. (Color online) Discrete diffraction in a $\mathcal{PT}$-waveguide array resulting from a single channel excitation.

this infinite array both the propagation shifts Δ_n and the coupling constant are now real and hence the modal amplitudes (associated with the bound states of the first band) evolve as if the array were entirely lossless [8], e.g., $i(\mathrm{d}a_n/\mathrm{d}\xi)+\kappa(a_{n+1}+a_{n-1})=0$. As a result, if the array is excited at the middle, the resulting discrete diffraction pattern follows the familiar Bessel distribution, that is, $a_n=(i)^n J_n(2\kappa\xi)$.

We would also like to stress that the Lagrangian formalism can also be used to obtain shifts in the propagation constant due to $\mathcal{PT}$ perturbations (i.e., perturbations that preserve the parity-time symmetry) below the phase transition. In this case we assume a solution of the form $\phi(\eta,\xi)=a(\xi)u(\eta)\exp(i\mu\xi)$, where $u(\eta)\exp(i\mu\xi)$ is the eigenfunction of the unperturbed system. Following exactly the same procedure as before and by solving the resultant differential equation we find

$$\Delta\mu = \frac{\int_{-\infty}^{\infty} u^*(-\eta)\Delta V(\eta)u(\eta)\mathrm{d}\eta}{\int_{-\infty}^{\infty} u^*(-\eta)u(\eta)\mathrm{d}\eta}, \tag{6}$$

where $\Delta\mu$ is the perturbation in the propagation constant due to the $\mathcal{PT}$ perturbation in the optical potential $\Delta V(\eta)$. We note that the result of Eq. (6) is again fundamentally different from that known from standard perturbation theory [9].

In conclusion, starting from Lagrangian principles, we have developed a formalism suitable for describing coupled optical $\mathcal{PT}$-symmetric systems.

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