Spectral renormalization method for computing self-localized solutions to nonlinear systems

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A new numerical scheme for computing self-localized states—or solitons—of nonlinear waveguides is proposed. The idea behind the method is to transform the underlying equation governing the soliton, such as a nonlinear Schrödinger-type equation, into Fourier space and determine a nonlinear nonlocal integral equation coupled to an algebraic equation. The coupling prevents the numerical scheme from diverging. The nonlinear guided mode is then determined from a convergent fixed point iteration scheme. This spectral renormalization method can find wide applications in nonlinear optics and related fields such as Bose–Einstein condensation and fluid mechanics. © 2005 Optical Society of America

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Optical spatial or temporal solitons in nonlinear media have attracted considerable attention in the scientific community. They have been demonstrated to exist in a wide range of physical systems in both continuous and discrete settings.[1–3] Such nonlinear modes can form as a result of a balance between diffraction or dispersion and nonlinearity. A central issue for these types of nonlinear guided waves is how to compute localized, i.e., soliton, solutions, which generally involve solving nonlinear ordinary or partial differential equations or difference equations. Various techniques have been used, e.g., shooting and relaxation techniques and the self-consistency method, to find nonlinear modes that utilize the concept that a soliton forms when the optical field induces a waveguide structure via the nonlinearity and self-traps itself (see, e.g., Refs. 4–6). Another method, introduced by Petviashvili,[7] for constructing localized solutions of a nonlinear system is based on transforming to Fourier space and determining a convergence factor based upon the degree (homogeneity) of a single nonlinear term (e.g., $|U|^p U$ has homogeneity $p+1$). While it was first used to find localized solutions in the two-dimensional Korteweg–deVries equation (usually referred to as the Kadomtsev–Petviashvili equation[8,9]), the method has been significantly extended and has been used to find localized solutions in a wide variety of interesting systems, e.g., dispersion-managed[10] and discrete diffraction-managed[11,12] nonlinear Schrödinger equations, dark and gray solitons,[13] and lattice vortices.[14] This method often is successful only when the underlying equation contains nonlinearity with fixed homogeneity. However, many physically interesting problems involve nonlinearities with different homogeneities, such as cubic–quintic, or even lack any homogeneity, as in saturable nonlinearity.

In this Letter we describe a novel spectral renormalization scheme with which we can compute localized solutions in nonlinear waveguides. The essence of the method is to transform the underlying equation that governs the soliton (e.g., nonlinear Schrödinger type) into Fourier space and find a nonlinear nonlocal integral equation (or system of integral equations) coupled to an algebraic equation (or system). The coupling prevents the numerical scheme from diverging. The nonlinear guided mode is then obtained from an iteration scheme, which in the cases we have investigated converges rapidly. The advantages of the present method are that (i) it can be applied to a large class of physically interesting problems including those in which the self-consistency method fails, as is the case for second-harmonic generation, (ii) it is relatively easy to implement (e.g., compared with relaxation methods), (iii) it can handle higher-order nonlinearities with different homogeneities, and (iv) it is spectrally efficient. Moreover, this method can find wide applications in nonlinear optics, Bose–Einstein condensation, and fluid dynamics. We begin by considering a scalar nonlinear Schrödinger-like equation:

$$i\frac{\partial U}{\partial z} + \nabla^2 U - V(\mathbf{x})U + N(|U|^2)U = 0, \tag{1}$$

where U is the envelope proportional to the electric field, z is the propagation direction, N is a nonlinearity that depends on intensity, $V(\mathbf{x})$ models an optical lattice, $\mathbf{x}=(x,y)$, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. A special class of soliton solution can be constructed by assuming that $U(\mathbf{x},z)=u(\mathbf{x};\mu)\exp(i\mu z)$, where μ is the propagation constant or the soliton eigenvalue. Substituting the above ansatz into Eq. (1), we get

$$-\mu u + \nabla^2 u - V(\mathbf{x})u + N(|u|^2)u = 0. \tag{2}$$

This is a nonlinear eigenvalue problem for u and μ that is supplemented with the boundary condition $u \to 0$ as $|r| \to +\infty$, where $r^2 = x^2 + y^2$. The scheme is based on Fourier analysis, which transforms Eq. (2) into a nonlocal equation that will then be solved using a convergent iteration. Define the Fourier transform $\mathcal{F}$ by

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$$\hat{u}(\mathbf{k}) = \mathcal{F}[u(\mathbf{x})] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(\mathbf{x}) \exp[i(k_x x + k_y y)] \mathrm{d}\mathbf{x}, \tag{3}$$

where $\mathrm{d}\mathbf{x}=\mathrm{d}x\mathrm{d}y$. First, consider the case with no optical potential or external defect ($V=0$), for which $\mu > 0$. Applying the Fourier transform to Eq. (2) leads to

$$\hat{u}(\mathbf{k}) = \frac{\mathcal{F}[N(|u|^2)u]}{\mu + |\mathbf{k}|^2}. \tag{4}$$

The idea is to construct a condition that limits the amplitude under iteration from either growing without bound or tending to zero. This is accomplished by introducing of a new field variable, $u(\mathbf{x}) = \lambda w(\mathbf{x})$, $\hat{u}(\mathbf{k}) = \lambda \hat{w}(\mathbf{k})$, where $\lambda \neq 0$ is a constant to be determined. Then the function $\hat{w}$ satisfies

$$\hat{w}(\mathbf{k}) = \frac{\mathcal{F}[N(|\lambda|^2|w|^2)w]}{\mu + |\mathbf{k}|^2} \equiv Q_\lambda[\hat{w}(\mathbf{k})]. \tag{5}$$

Multiplying Eq. (5) by $\hat{w}^*(\mathbf{k})$ and integrating over the entire $\mathbf{k}$ space, we find the relation

$$G(\lambda) \equiv \int_{-\infty}^{+\infty} |\hat{w}(\mathbf{k})|^2 \mathrm{d}\mathbf{k} - \int_{-\infty}^{+\infty} \hat{w}^*(\mathbf{k}) Q_\lambda[\hat{w}(\mathbf{k})] \mathrm{d}\mathbf{k} = 0, \tag{6}$$

providing an algebraic condition on the constant λ. Finally, the desired solution is obtained by iterating Eq. (5):

$$\hat{w}_{m+1}(\mathbf{k}) = \frac{\mathcal{F}[N(|\lambda_m|^2|w_m|^2)w_m]}{\mu + |\mathbf{k}|^2} \tag{7}$$

for $m \geqslant 1$; λ_m denotes the solution to $G(\lambda_m)=0$ at iteration m. Other variants of Eq. (6) are possible. But what is crucial is to solve Eq. (7) coupled to an algebraic-type equation that is obtained from Eq. (5) by imposing an integral identity such as Eq. (6). We name this method spectral renormalization. It is straightforward to implement Eq. (7): Initially we guess a function $w_1(\mathbf{x})$ [e.g., a Gaussian or sech-like profile], which from Eq. (6) yields λ_1 that satisfies $G(\lambda_1)=0$. Then, from Eq. (7), we obtain $\hat{w}_2(\mathbf{k})$, which from the inverse, Fourier transform leads to $w_2(\mathbf{x})$. The iteration continues until convergence is achieved. The procedure described above reproduces the results presented elsewhere.[12] Knowing the weakly nonlinear limit is very useful in this regard. Before presenting specific examples, we explain how to construct localized solutions in the presence of an external defect ($V \neq 0$). In this case, $\mu < 0 (\mu \equiv -|\mu|)$. But dividing by the expression $|\mu| - |\mathbf{k}|^2$ leads to a singularity. To avoid this, we add to and subtract from Eq. (2) the term $ru(\mathbf{x})$, where r is a positive constant, and then take the Fourier transform. This leads to

$$\hat{u}(\mathbf{k}) = \frac{(r+|\mu|)\hat{u}}{r + |\mathbf{k}|^2} - \frac{\mathcal{F}[Vu] - \mathcal{F}[N(|u|^2)u]}{r + |\mathbf{k}|^2} \equiv R[\hat{u}(\mathbf{k})]. \tag{8}$$

Following the change of variable $u(\mathbf{x}) = \lambda w(\mathbf{x})$ the iteration scheme takes the form $\hat{w}_{m+1}(\mathbf{k}) = R[\lambda_m \hat{w}_m(\mathbf{k})]$, with λ_m given by the relation

$$\int_{-\infty}^{+\infty} |\hat{w}_m(\mathbf{k})|^2 \mathrm{d}\mathbf{k} - \int_{-\infty}^{+\infty} \hat{w}_m^*(\mathbf{k}) R[\lambda_m \hat{w}_m(\mathbf{k})] \mathrm{d}\mathbf{k} = 0. \tag{9}$$

To illustrate the method in a prototypical problem we consider photorefractive lattice solitons in self-focusing saturable nonlinearity, for which $V(\mathbf{x}) = I_0[\cos^2(x) + \cos^2(y)]$ and $N(|u|^2) = -1/(1+|u|^2)$. These photorefractive solitons were observed experimentally for the first time by Segev's group.[15] The lattice modes were originally found using the self-consistency method.[16] The reason for our showing this example is to delineate a situation in which the nonlinearity does not have a well-defined homogeneity. For the fully saturable case, the iteration scheme reads as

$$\hat{w}_{m+1}(\mathbf{k}) = \frac{(r+|\mu|)}{r+|\mathbf{k}|^2}\hat{w}_m - \frac{\mathcal{F}[Vw_m]}{r+|\mathbf{k}|^2} + \frac{1}{r+|\mathbf{k}|^2}\mathcal{F}\left[\frac{w_m}{1+|\lambda_m|^2|w_m|^2}\right], \tag{10}$$

where λ_m are obtained from iterating Eq. (9) by using standard nonlinear algebraic equation solvers, e.g., the Newton method. A typical example of a fundamental discrete soliton that corresponds to the parameters $I_0=1$ and $\mu=0.8$ is shown in Fig. 1. We have verified the stationarity of the mode by using direct numerical simulation in Eq. (1). We can readily generalize the above method to include more than one field. In that case, Eq. (2) is replaced by M coupled stationary nonlinear Schrödinger-like equations that are solvable using the same idea outlined above.

Another interesting case that arises in many applications is that of second-harmonic generation. The system of equations governing stationary soliton states[17–21] is

$$-\nu A + W(\mathbf{x})A + \nabla^2 A + AB = 0, \tag{11}$$

$$-4\nu B + 4W(\mathbf{x})B + \nabla^2 B + \frac{A^2}{2} = 0, \tag{12}$$

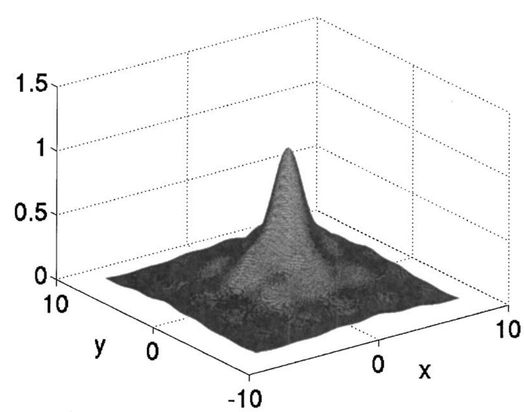

Fig. 1. Fundamental lattice soliton obtained by iterating Eq. (10) for $I_0=1$ and $|\mu|=0.8$.

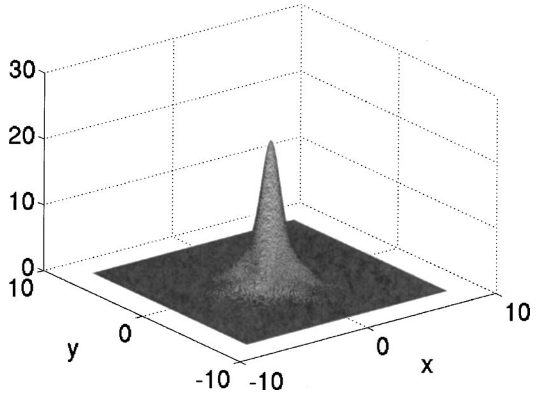

Fig. 2. Fundamental harmonic $|A|$ for $W_0=2$ and $\nu=2$.

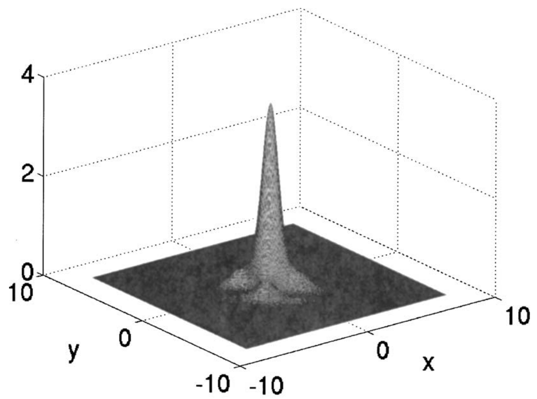

Fig. 3. Second harmonic $|B|$ for $W_0=2$ and $\nu=2$.

where $\nu>0$ is the soliton propagation constant and the lattice potential is given by $W(\mathbf{x}) = W_0 \cos(2x)\cos(2y)$. Following a similar procedure, we adopt the change of variables $A=\lambda_1\psi$, $B=\lambda_2\phi$. In this case the iteration scheme takes the form

$$\hat{\psi}_{m+1} = \frac{\mathcal{F}[W\psi_m] + \lambda_{2m}\mathcal{F}[\psi_m\phi_m]}{\nu + |\mathbf{k}|^2}, \tag{13}$$

$$\hat{\phi}_{m+1} = \frac{4\mathcal{F}[W\phi_m] + (\lambda_{1m}^2/\lambda_{2m})\mathcal{F}[\psi_m^2/2]}{4\nu + |\mathbf{k}|^2}. \tag{14}$$

The convergence factors λ_{1m} and λ_{2m} satisfy the coupled system

$$\lambda_{2m} = -\frac{\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}[(\nu + |\mathbf{k}|^2)|\hat{\psi}_m|^2 - \hat{\psi}_m^*\mathcal{F}[W\psi_m]]}{\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\,\hat{\psi}_m^*\mathcal{F}[\psi_m\phi_m]},$$

$$\lambda_{1m}^2 = \frac{\lambda_{2m}\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}[(4\nu + |\mathbf{k}|^2)|\hat{\phi}_m|^2 - 4\hat{\phi}_m^*\mathcal{F}[W\phi_m]]}{\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\,\hat{\phi}_m^*\mathcal{F}[\psi_m^2/2]}.$$

Typical examples of quadratic lattice solitons are shown in Figs. 2 and 3.

In conclusion, we have developed a novel numerical scheme with which to compute self-localized states of nonlinear waveguides that is flexible and can be applied to many nonlinear systems. As prototypical examples, we considered photorefractive saturable nonlinearity, which lacks the property of homogeneity and second-harmonic generation. We have shown how to find lattice solitons by using this spectral renormalization method.

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