Quantum phase distribution of thermal phase-squeezed states

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The quantum phase distributions and variances of strong coherent and phase-squeezed states mixed with thermal light are calculated. We find that the effect of thermal light on the coherent phase distribution becomes important only when the number of thermal photons is of order one-half. Its effect on the phase distribution of phase-squeezed states (for the case where the number of photons is very large compared to e^{2r}) becomes important when the number of thermal photons is of order $\frac{1}{2}e^{-2r}$, where r is the squeezing parameter. [S1050-2947(98)07901-3]

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As is well known, the nonclassical properties of electromagnetic (em) waves are destroyed by the presence of noise and losses. The influence of thermal noise on several nonclassical properties of squeezed states has been analyzed in previous works [1–5]. However, the effect of thermal noise on quantum phase measurement, especially for nonclassical light, has attracted only a little attention. The aim of the present article is to study such effects. We analyze the effect of thermal noise on phase measurements of strong em fields (for which the quantum phase is well defined) and include in our analysis in particular the coherent and squeezed states. We study the extent to which the thermal noise destroys the nonclassical phase properties of phase-squeezed states.

We shall follow the Glauber-Lachs formalism [6–8] in quantum optics for describing admixtures of thermal and coherent radiation. By a direct generalization of this approach, the admixture of the pure density $\hat{\rho}_0$ with thermal light in the same mode is given by [1,4,5]

$$\hat{\rho}(\overline{n}) = \int \frac{d^2\beta}{\pi \overline{n}} \exp(-|\beta|^2/\overline{n}) \hat{D}(\beta) \hat{\rho}_0 \hat{D}^\dagger(\beta), \tag{1}$$

where $\hat{D}(\beta)$ is the standard displacement operator and $\overline{n}$ is the mean number of thermal photons ''added'' to the one-mode em field. The physical meaning of this formula has been discussed in a previous work [5]. In the present article we would like to relate the general formula (1) to the effect of thermal light on quantum phase measurements of coherent and phase-squeezed states. A pure displaced squeezed state is described by the density operator

$$\hat{\rho}_0 = |\alpha,\xi\rangle\langle\alpha,\xi|, \tag{2}$$

where

$$|\alpha,\xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle \tag{3}$$

is the displaced squeezed state,

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \tag{4}$$

is the usual displacement operator, and

$$\hat{S}(\xi) = \exp[\tfrac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})] \tag{5}$$

is the squeezing operator with the squeezing parameter $\xi = re^{i\phi}$. Throughout the present article we will take $\phi = 0$ and real α. The Wigner function corresponding to the density operator of Eq. (1) with the density operator $\hat{\rho}_0$ of displaced squeezed states is given by [5,9]

$$W_{\mathrm{TSS}}(\overline{n}) = \frac{1}{\pi\sqrt{(2\overline{n}+e^{2r})(2\overline{n}+e^{-2r})}} \times \exp\left(-\frac{(x-\sqrt{2}\alpha)^2}{2\overline{n}+e^{2r}} - \frac{p^2}{2\overline{n}+e^{-2r}}\right), \tag{6}$$

where the subscript TSS denotes thermal squeezed state.

In the present work we calculate the Wigner phase distribution of a phase-squeezed state mixed with thermal noise. For this purpose we express the Wigner function of Eq. (6) in polar coordinates and integrate over the radial variable. In this way we generalize the procedure used in a previous work [10] to treat here the effects of thermal light on the phase distribution. Since we are interested especially in strong em fields, the present procedure gives results similar to those that can be obtained by other methods [10–13], but it is more convenient for an analytical treatment.

The phase distribution of displaced squeezed states mixed with thermal noise is given by

$$W_{\mathrm{TSS}}(\theta) = \int_0^\infty \xi W_{\mathrm{TSS}}(x=\xi\cos\theta, p=\xi\sin\theta)\, d\xi. \tag{7}$$

By substituting Eq. (6) into Eq. (7) we get after straightforward integration

$$W_{\mathrm{TSS}}(\theta)=\frac{e^{-c}}{2\pi a\sqrt{(2\bar{n}+e^{2r})(2\bar{n}+e^{-2r})}}\times\left[\frac{b}{2}\sqrt{\frac{\pi}{a}}e^{b^2/4a}\operatorname{erfc}\left(\frac{-b}{2\sqrt{a}}\right)+1\right], \qquad (8)$$

where

$$a=\frac{2\bar{n}+e^{-2r}\cos^2\theta+e^{2r}\sin^2\theta}{(2\bar{n}+e^{-2r})(2\bar{n}+e^{2r})},$$

$$b=\frac{2\sqrt{2}\alpha\cos\theta}{2\bar{n}+e^{2r}},$$

$$c=\frac{2\alpha^2}{2\bar{n}+e^{2r}}.$$

The Wigner phase distribution for thermal coherent states is obtained from Eq. (8) by substituting $r=0$, whereas for pure phase-squeezed light by $\bar{n}=0$. Although Eq. (8) describes a very general result, we would like to concentrate here on strong em fields for which we can use the approximations $\alpha \gg e^{2r}$, $\bar{n}\ll\alpha^2$, and $\theta\ll 1$. These approximations have a simple geometrical interpretation [10] in the Wigner phase-space description. The approximation $\alpha\gg e^{2r}$ means that the width of the Wigner function is very small relative to the distance of its center from the origin and consistently the approximation $\theta\ll 1$ means that the opening angle for the phase distribution is very small [10]. The thermal noise leads to smearing of the phase distribution, but for $\bar{n}\ll\alpha^2$ the above approximations are still valid. Under the above-mentioned approximations $b^2\gg 4a$, so that we can neglect the term 1 in the square brackets of Eq. (8) and use the approximation $\operatorname{erfc}(-b/2\sqrt{a})\simeq 2$. Using these approximations, we obtain the result

$$W_{\mathrm{TSS}}(\theta)\simeq\sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{2\bar{n}+e^{-2r}}}\exp\left(-\frac{2\alpha^2\theta^2}{2\bar{n}+e^{-2r}}\right). \qquad (9)$$

The variance of the phase distribution of phase-squeezed states mixed with thermal noise is obtained approximately as

$$(\Delta\theta)^2_{\mathrm{TSS}}=\frac{2\bar{n}+e^{-2r}}{4N}, \qquad (10)$$

where $N=\alpha^2$ is the number of coherent photons. Notice that for $\bar{n}=0$ and $r=0$ we recover the well-known result of $(\Delta\theta)^2_{\mathrm{coh}}=1/4N$. The variance of the phase distribution of thermal phase-squeezed states can be calculated also from the geometrical picture of the phase opening in the Wigner phase space

$$\begin{aligned}(\Delta\theta)^2_{\mathrm{TSS}}=\frac{(\Delta p)^2_{\mathrm{TSS}}}{2\alpha^2}&\simeq-\frac{1}{4\pi\bar{n}\alpha^2}\int d^2\beta\; e^{-|\beta|^2/\bar{n}}\\&\times\langle 0|\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)(\hat{a}-\hat{a}^\dagger+2i\beta_y)^2\hat{D}(\alpha)\hat{S}(\xi)|0\rangle\\&+\frac{1}{4\pi\bar{n}\alpha^2}\left[\int d^2\beta\; e^{-|\beta|^2/\bar{n}}\right.\\&\left.\times\langle 0|\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)(\hat{a}-\hat{a}^\dagger+2i\beta_y)\hat{D}(\alpha)\hat{S}(\xi)|0\rangle\right]^2\\&=\frac{2\bar{n}+e^{-2r}}{4N}. \end{aligned} \qquad (11)$$

More accurate results for the phase distribution of thermal phase-squeezed states can be made by numerical calculations using Eq. (8). The effect of squeezing in Eqs. (9) and (10) is to decrease the width of the quantum phase distribution while the thermal noise makes it broader. The effect of thermal noise for phase-squeezed states becomes important for values of $2\bar{n}$ that are of order e^{-2r}. We find that for a laser radiation at about 10μ and at room temperature, the number of thermal photons is approximately $e^{-4.3}$ [11]. The effect of thermal photons on the phase distribution of coherent states under these conditions is negligible. However, for phase-squeezed radiation the effect of thermal noise becomes important for $r\gtrsim 1.8$. Increasing the squeezing parameter beyond this limit will not improve the phase measurement significantly due to admixture of thermal light. Therefore, the effect of thermal noise is much more important for phase-squeezed states than for the phase distribution of coherent states, where its effect becomes important only for values of $2\bar{n}$ that are of order unity. In conclusion, we have treated the effect of thermal noise on the phase distribution for strong em fields for which the phase is well defined. We have not treated this topic for weak em fields for which there are various theoretical problems in the description of the phase distribution [12–17].

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