Methods for discrete solitons in nonlinear lattices

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A method to find discrete solitons in nonlinear lattices is introduced. Using nonlinear optical waveguide arrays as a prototype application, both stationary and traveling-wave solitons are investigated. In the limit of small wave velocity, a fully discrete perturbative analysis yields formulas for the mode shapes and velocity.

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In recent years, there has been considerable interest in the study of discrete spatial solitons in nonlinear media (cf. [1]). Such solitons are intrinsic highly localized modes of nonlinear lattices [2]. In a coupled optical waveguide, they form when a beam of high intensity locally changes the nonlinear refractive index of the waveguide via the Kerr effect and decouples them from the remaining waveguides. In this case, discrete diffraction is balanced by nonlinearity.

Discrete solitons have been demonstrated to exist in a wide range of physical systems, e.g., atomic chains [3,4] (discrete lattices) with on-site cubic nonlinearities, molecular crystals [5], biophysical systems [6], electrical lattices [7], and recently in arrays of coupled nonlinear optical wave guides [8,9]. In all of the above cases, the localized modes are solutions of the well-known discrete nonlinear Schrödinger (DNLS) equation. The DNLS as a model of nonlinear optical waveguide arrays was first suggested by Christodoulides and Joseph [2] and later various applications of discrete solitons were explored, e.g., storage and steering of discrete solitons in waveguide array [10].

The first experimental observation of discrete solitons in optical waveguide arrays was reported in [8]. When a low-intensity beam was injected to one waveguide, the propagating field spreads over the adjacent waveguides, hence experiencing discrete diffraction. However, for sufficiently high power, the beam was self trapped in the central waveguide. Subsequently, the dynamical behavior of discrete solitons was experimentally observed [9].

In this paper, we introduce a method to obtain both stationary and moving solitons in nonlinear lattices. The essence of the method is to transform the DNLS equation governing the solitary wave into Fourier space, where the wave function is smooth, and then deal with a nonlinear nonlocal integral equation for which we employ a rapidly convergent numerical scheme to find solutions. A key advantage of the method is to transform a differential-delay equation into an integral equation for which computational methods are effective (see also Refs. [11,12]). Importantly, the technique allows us to explore physical phenomena such as discrete solitons in a nonlinear waveguide array with varying diffraction [13]. This relates to the recent experimental observation of diffraction management in optical waveguides [14]. Mathematically, the method also provides a foundation upon which an analytic theory describing solitons in nonlinear lattices can be constructed. Applying this method to the DNLS model, shows that traveling solitons possess a nontrivial nonlinear "chirp." Moreover, our results (both numerical and analytical) indicate that, unlike the integrable case [15], *a continuous traveling-wave solution may not exist* [16]. In the limit of small velocity, we develop a fully discrete perturbation theory and show that slowly moving discrete solitons are indeed "chirped."

Consider an infinite array of one-dimensional identical waveguides with equal separation. The equation, which models the evolution of the slowly varying envelope of the electric field, is the well-known DNLS equation,

$$i\frac{\partial\phi_n}{\partial z}+\frac{1}{h^2}(\phi_{n+1}+\phi_{n-1}-2\phi_n)+|\phi_n|^2\phi_n=0, \qquad (1)$$

where ϕ_n is the on-site wave function, h is the lattice spacing, and z is the propagation distance. Another important model for discrete solitons is the integrable DNLS equation (IDNLS) [15] in which the nonlinearity takes the average form $|\phi_n|^2(\phi_{n+1}+\phi_{n-1})/2$. We look for traveling localized modes in the form

$$\phi_n(z)=u(\xi)e^{-i\psi_n}, \qquad (2)$$

with $\xi=nh-Vz$ and $\psi_n=\beta nh-\omega z$ where V and ω are the soliton velocity and wave-number shift, respectively. Assuming u is complex, i.e., $u(\xi)=F(\xi)+iG(\xi)$ (with F,G being real), then Eq. (1) takes the form

$$\begin{aligned} &VG'+\mathcal{D}_1F+\mathcal{D}_2G+(F^2+G^2)F=\omega F,\\ &-VF'+\mathcal{D}_1G-\mathcal{D}_2F+(F^2+G^2)G=\omega G, \end{aligned} \qquad (3)$$

where prime denotes derivative with respect to ξ and

$$\begin{aligned} &\mathcal{D}_1\mathcal{F}=\frac{1}{h^2}[\cos(\beta h)(E_++E_-)\mathcal{F}-2\mathcal{F}],\\ &\mathcal{D}_2\mathcal{G}=\frac{\sin(\beta h)}{h^2}(E_+-E_-)\mathcal{G}, \end{aligned} \qquad (4)$$

with $E_\pm X(\xi)\equiv X(\xi\pm h)$. To solve system (3), i.e., to find the mode shapes and the velocity dependence on β, we use discrete Fourier analysis. The advantage is that the differential delay Eq. (3) is transformed into a nonlinear nonlocal integral equation in which the derivatives in Eq. (3) are replaced by polynomials. Having transformed the equation into an integral equation, the soliton is then viewed as a fixed point of a nonlinear functional.

Stationary solitons. Stationary solutions are obtained by solving Eq. (3) for $V=\beta=0$, $\omega=\omega_s$, and $G=0$. We introduce the discrete Fourier transform

$$\hat{F}(q)=\sum_{m=-\infty}^{+\infty} F(mh)e^{-iqmh},$$
$$F(mh)=\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}\hat{F}(q)e^{iqmh}\,dq. \tag{5}$$

Note the difference from the continuous Fourier transform where the sum is replaced by an integral and $h\rightarrow 0$ in the inverse transform. Then, Eq. (3) transforms into the following nonlinear integral equation:

$$\hat{F}(q)=\frac{h^2}{4\pi^2\Omega(q)}(\hat{F}*\hat{F}*\hat{F})(q)\equiv\mathcal{K}_{\omega_s}[\hat{F}(q)],$$
$$\hat{F}*\hat{F}*\hat{F}=\int dq_1\,dq_2\hat{F}(q_1)\hat{F}(q_2)\hat{F}(q-q_1-q_2), \tag{6}$$

and $\Omega(q)=\omega_s+2[1-\cos(hq)]/h^2$ is the frequency of the linear excitations. Equation (6) bears the following important conclusion: solitons can be viewed as being a fixed point of an infinite-dimensional nonlinear integral equation. To numerically find the fixed point, we employ a modified Neumann iteration scheme (cf. [11,12]) and write Eq. (6) in the form

$$\hat{F}_{n+1}(q)=\left(\frac{\alpha(\hat{F}_n)}{\beta(\hat{F}_n)}\right)^{3/2}\mathcal{K}_{\omega_s}[\hat{F}_n(q)],\quad n\geqslant 0,$$
$$\alpha=\int\hat{F}_n^2(q)dq;\quad \beta=\int\hat{F}_n(q)\mathcal{K}_{\omega_s}[\hat{F}_n(q)]dq. \tag{7}$$

The factor 3/2 is chosen to make the right hand side of Eq. (7) of degree zero, which yields convergence of the scheme [11,12]. When $F(mh)$ is real and even, it implies that $\hat{F}(q)$ is also real. Clearly when $\hat{F}_n(q)\rightarrow\hat{F}_s(q)$ as $n\rightarrow\infty$, then $\alpha/\beta\rightarrow 1$ and in turn $\hat{F}_s(q)$ will be the solution to Eq. (6). The factors α and β are introduced to stabilize an otherwise divergent simple Neumann iteration scheme. Note that when we apply the continuous Fourier transform on Eqs. (3) (to find stationary solution), then the numerical scheme based on Eq. (7) does not converge, which indicates that a *continuous* stationary solution to the DNLS may not exist. As we will see later, this will have a direct impact on the traveling-wave problem. Figure 1 shows a typical solution to Eq. (6) both in the Fourier domain [Fig. 1(a)] and in physical space [Fig. 1(b)] for different values of lattice spacing h. Importantly, with suitable modification of Eq. (7), this method yields breathing localized modes: "discrete diffraction managed solitons" [13].

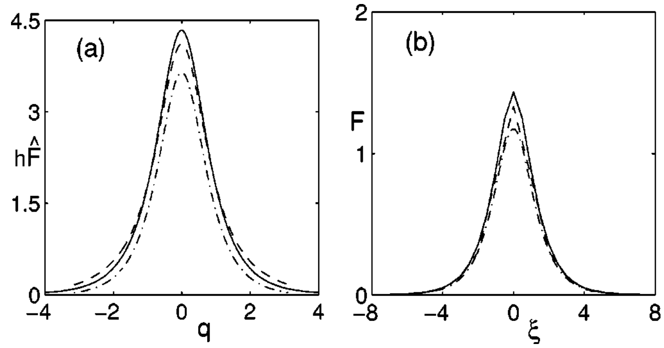

FIG. 1. Mode profiles obtained with $\omega_s=1$ in Fourier space (a), for $h=0.5$ (solid), $h=1$ (dashed), and $h=1$ (dashed dotted) for the integrable case. (b) Soliton shape in physical space for $h=0.5$ (solid), $h=1$ (dashed), and for the integrable case at $h=1$ (dashed dotted).

Traveling solitons. Unlike the IDNLS in which traveling solitons are exact and continuous solutions, there are no known explicit solutions for DNLS solitons. Previous studies of traveling waves employed various techniques [17,18] and ansatz based on stationary solutions and perturbation of IDNLS [19]. Our analysis, which is based on the discrete Fourier methods, reveals another fundamental distinction from the IDNLS traveling solitons: they are "multimode" discrete solitons, i.e., a single mode (sech-like shape) does not propagate without significant radiation. Importantly, when we apply the *continuous* Fourier transform on Eqs. (3), we find that the numerical scheme based on Eqs. (7) and (8) with $\pi/h\rightarrow\infty$ does not converge to a solution. This is a strong indication that, as opposed to the integrable case, a *true continuous* stationary or traveling-wave solutions to the DNLS model does not exist. By continuous solution, we mean a solution that can be defined off the lattice points, which is *necessary* when discussing traveling waves on lattices. In fact, the perturbation analysis presented below supports this observation as it *fails* to give consistent results off the grid points. Similarly, it was shown in [20] that the NLS equation with a positive fourth-order correction (which is obtained from the DNLS by taking the limit $h\rightarrow 0$) lacks exact soliton solutions. These results differ from those of [21] in which a "continuous" traveling solitary waves were reported using Fourier series expansions with finite period L while *assuming convergence* as $L\rightarrow\infty$. To find the mode shapes and soliton velocity, we proceed as before by taking the discrete Fourier transform of Eqs. (3), which yields the following iteration scheme:

$$\hat{F}_{n+1}(q)=\frac{\Omega_2(q)}{\Omega_1(q)}\tilde{G}_n(q)+\left(\frac{\alpha_1}{\beta_1}\right)^{3/2}\mathcal{Q}_1[\hat{F}_n,\tilde{G}_n],$$
$$\tilde{G}_{n+1}(q)=\frac{\Omega_2(q)}{\Omega_1(q)}\hat{F}_n(q)+\left(\frac{\alpha_2}{\beta_2}\right)^{3/2}\mathcal{Q}_2[\hat{F}_n,\tilde{G}_n], \tag{8}$$

where $\hat{F}(q)$ and $\hat{G}(q)\equiv -i\tilde{G}(q)$ are the Fourier transforms of $F(\xi)$ and $G(\xi)$, respectively; and

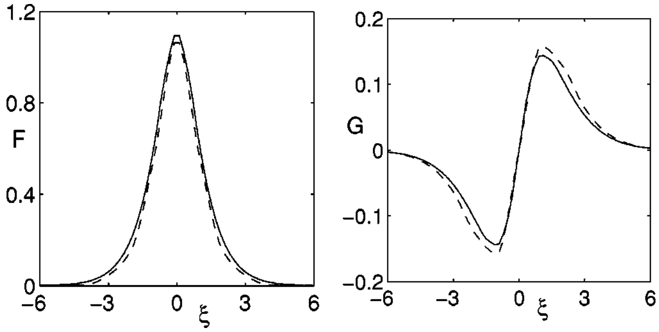

FIG. 2. Mode shapes in physical space for $\omega=1$ and $\beta=0.5$. Solid line corresponds to $h=0.5$ and velocity $V=-0.25$ whereas dashed line for $h=1$ and $V=-0.155$.

$$\mathcal{Q}_1[\hat{F},\hat{G}]=\frac{h^2}{4\pi^2\Omega_1(q)}(\hat{F}*\hat{F}*\hat{F}-\tilde{G}*\tilde{G}*\hat{F}),$$
$$\mathcal{Q}_2[\hat{F},\tilde{G}]=\frac{h^2}{4\pi^2\Omega_1(q)}(\hat{F}*\hat{F}*\tilde{G}-\tilde{G}*\tilde{G}*\tilde{G}). \tag{9}$$

The convergence factors α_j and β_j, $j=1,2$ are given by

$$\alpha_1=\int \hat{F}_n(q)\left(\hat{F}_n(q)-\frac{\Omega_2(q)}{\Omega_1(q)}\tilde{G}_n(q)\right)dq$$
$$\beta_1=\int \hat{F}_n(q)\mathcal{Q}_1\,dq, \quad \beta_2=\int \tilde{G}_n(q)\mathcal{Q}_2\,dq, \tag{10}$$

and α_2 is obtained from α_1 by interchanging $\hat{F}$ and $\hat{G}$. Here, $\Omega_1(q)=\omega+2/h^2[1-\cos(hq)\cos(\beta h)]$ and $\Omega_2(q)=2/h^2\sin(hq)\sin(\beta h)+Vq$. For a given set of parameters h, ω and $\beta>0$, the mode shapes and soliton velocity are found by iterating Eqs. (8) with an initial guess, e.g., $\hat{F}_0(q)=\mathrm{sech}(q)$, $\tilde{G}_0(q)=\mathrm{sech}(q)\tanh(q)$ and $V=V_*<0$. The iterations are carried out until the condition $|\mathcal{E}_j|\equiv|\alpha_j-\beta_j|<\epsilon$, $(j=1,2)$ is satisfied with $\epsilon>0$ being a prescribed tolerance. However, unlike the stationary case, here, the soliton velocity is still to be determined. For any choice of $V_*<0$ if $|\mathcal{E}_j|\nless\epsilon$, we seek a different value of V_* at which $\mathcal{E}_j$ changes sign. Then, we use the bisection method to change V_* in order to locate the correct velocity V and modes $\hat{F},\tilde{G}$ for each ω, β, and h. Typical soliton modes are shown in Fig. 2.

Although Eqs. (8) can be solved numerically with high accuracy, the resulting solutions are only obtained at the discrete locations $\xi=nh$, while all real values of ξ are called upon in a traveling-wave solution. So the question we want to ask is: What happens to the modes found above when they propagate across the array? To answer this question, we simulated Eq. (1) using $\phi_n(z=0)=u(nh)e^{-i\beta nh}$ as an initial condition with $u(nh)=F_{\mathrm{TW}}(nh)+iG_{\mathrm{TW}}(nh)$ being the solutions obtained from Eq. (8). When a moderately localized mode [22] is launched, the beam moved across the waveguides undistorted (Fig. 3) over 100 normalized z units. This corresponds, according to the experimental data reported in [8], to 120 mm (recall that the wave guides used in [8] were 6 mm in length). On the other hand, strongly localized modes travel essentially undistorted for shorter distances (around 20 normalized z units, see Fig. 4) which corresponds to 24 mm. Noticeably, during propagation, there was a change of 0.0133%/mm (0.245%/mm) in the soliton velocity for moderately (strongly) localized modes, in which case strongly localized mode slows down and eventually relaxes to a stationary state. This behavior depends crucially on the initial amplitude: Higher-amplitude solitons are less "mobile" than lower-amplitude beams. The discrete Fourier transform yields a useful, but nonuniform traveling-wave solution. We term this a "*stroboscopic*" traveling wave.

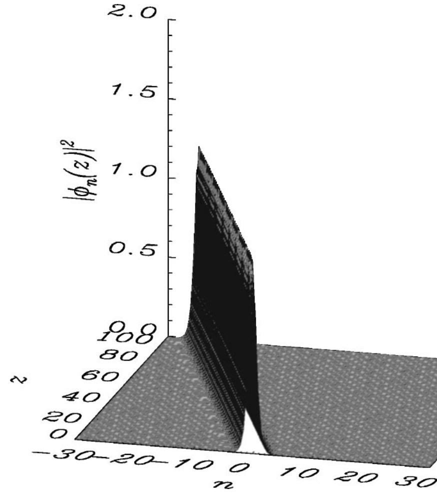

FIG. 3. Evolution of a moderately localized soliton in physical domain for $\beta=0.5$, $V=-0.25$, $\omega=1$, and $h=0.5$ obtained by direct numerical simulation.

Perturbation theory. To analytically explore the multimode nature of the DNLS traveling solitons, we consider the case in which the solitons move slowly and develop a fully discrete perturbation theory for finite amplitude. It is important to note that our perturbative approach is fundamentally different than the perturbation methods based on inverse scattering theory (cf. [19,23]).

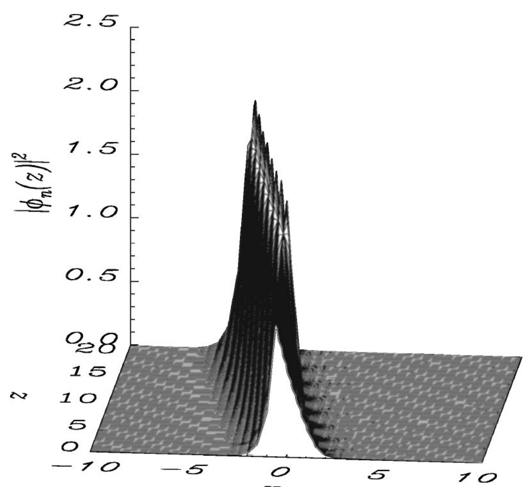

FIG. 4. Evolution of a strongly localized soliton in physical domain for $\beta=0.5$, $V=-0.2$, $\omega=2$, and $h=0.5$ obtained by direct numerical simulation.

Next, we take $\beta=\epsilon\beta_1+O(\epsilon^2)$, $\epsilon\ll1$, and expand the soliton velocity, frequency, and the wave functions in a power series in ϵ. Keeping terms up to $O(\epsilon^3)$ we have

$$V=\epsilon V_1+\epsilon^2 V_2,\quad \omega=\omega_s+\epsilon\omega_1+\epsilon^2\omega_2,$$
$$F=F_0+\epsilon F_1+\epsilon^2 F_2,\quad G=\epsilon G_1+\epsilon^2 G_2. \tag{11}$$

Substituting Eqs. (11) into (3), we find F_0 satisfies the stationary equation and is even in ξ and to order ϵ

$$\mathcal{L}_1 F_1=\omega_1 F_0(\xi),$$
$$\mathcal{L}_2 G_1=V_1 F_0'(\xi)+\beta_1(E_+-E_-)F_0(\xi)/h, \tag{12}$$

and to order ϵ^2,

$$\mathcal{L}_1 F_2=\omega_1 F_1+\omega_2 F_0-V_1 G_1'(\xi)-F_0 G_1^2-3F_0F_1^2$$
$$+\beta_1^2(E_++E_-)F_0/2-\beta_1(E_+-E_-)G_1/h, \tag{13}$$

$$\mathcal{L}_2 G_2=\omega_1 G_1+V_1F_1'(\xi)+V_2F_0'(\xi)-2F_0F_1G_1$$
$$+\beta_1(E_+-E_-)F_1, \tag{14}$$

where $\mathcal{L}_j X=-\omega_s X+(E_++E_--2)X+l_jF_0^2X$, $j=1,2$ and $l_1=3$, $l_2=1$. Solutions to system (12) are given by

$$F_1=\omega_1\partial F_0/\partial\omega_s+b\,\partial F_0/\partial\xi,$$
$$G_1=V_1A+\beta_1\xi F_0+cF_0, \tag{15}$$

where b and c are arbitrary constants and $\mathcal{L}_2A=\partial F_0/\partial\xi$, which can be solved by Fourier transform method. The velocity $V_1(\beta_1)$ and frequency shift ω_1, are determined by a solvability condition at order ϵ^2 which is the discrete analog of Green's identity. For ξ restricted to the grid points, i.e., $\xi=\xi_l\equiv lh$ (which is consistent with the discrete Fourier transform), we find that $\omega_1=0$ and $V_1=-\gamma(h)\beta_1$; $\gamma(h)\equiv a_1(h)/a_2(h)$ where

$$a_1=\sum_{l\in\mathbb{Z}}F_0'(\xi_l)[2\xi_lF_0^3(\xi_l)+\tfrac{1}{h}(E_+-E_-)F_0(\xi_l)],$$
$$a_2=\sum_{l\in\mathbb{Z}}2F_0'(\xi_l)A(\xi_l)F_0^2(\xi_l)+F_0'^2(\xi_l). \tag{16}$$

We compared these semianalytical results with direct numerical simulation for the fully discrete case and found excellent agreement. Moreover, in the limit $h\to0$, we retrieve the known result $V_1=-2\beta_1$ and $G_1(\xi)\to0$.

In conclusion, we have proposed a method, based on discrete Fourier transforms, to compute stationary and traveling soliton in nonlinear lattices. This method can be applied to a wide variety of problems in nonlinear dynamics of discrete systems. It also provides a foundation for a rigorous theory in which fixed point theorems can be developed. Applying this method to the DNLS model, shows that traveling solitons have "multimode" structure. Our findings agree with direct numerical simulations and are consistent with perturbation theory, which yields explicit formulas for the soliton modes and velocity.

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