ELSEVIER

Physica A 249 (1998) 141–145

PHYSICA A

Long-wave instability in optical parametric oscillators

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Abstract

We show that the full dynamical equations for optical parametric oscillators in a large aspect ratio cavity for the case of long-wave instability is reduced to a Ginzburg–Landau equation near the instability threshold. This equation enables us to introduce the concept of optical vortices in $\chi^{(2)}$ medium. A criterion of supercritical and subcritical instability is given as well as a condition of the Benjamin–Feir instability. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Pattern formation in nonlinear dynamical systems has recently attracted considerable attention from the scientific community [1]. Of particular interest is the study of complex structures in nonlinear optics resulting from the appearance of different kinds of instabilities [2,3]. The physical origin of these instabilities, in the case of optical parametric oscillators, is the coupling of diffraction with an optical $\chi^{(2)}$ nonlinearity in two transverse spatial dimensions.

An increasing interest has been focused on large aspect ratio nonlinear optical systems, where pattern formation is independent of transverse boundaries and is thus described by a universal order parameter equations which provide a connection to other nonlinear behaviours. Transverse pattern formation for large aspect ratio optical parametric oscillators and lasers has been studied for cavity configurations with flat-end mirrors of infinite transverse extension and uniform pumping. In the latter case, it was shown [4–6] that the full Maxwell–Bloch equations admit exact travelling wave solutions. Moreover, the nature of the solutions above threshold strongly depends on the sign of the detuning parameter.

Pattern formation in optical parametric oscillators (OPO) has been intensively studied for the degenerate [7] as well as for the nondegenerate cases. Analytical and numerical studies in the case of degenerate OPO has shown that roll patterns organize the spatio-temporal evolution of OPO dynamics. For the nondegenerate case it was recently

PII S0378-4371(97)00445-7

shown that the full OPO dynamical equations admit a continuum family of stable travelling waves which are preferred to standing waves found in the degenerate case. The analysis of stability of travelling wave against standing wave solutions was investigated by deriving two coupled Newell–Whitehead–Segel equations describing the growth of both solutions close to threshold [8].

Quite recently [9], a Swift–Hohenberg equation was derived for a single longitudinal mode OPO operating near resonance for the case of small detuning parameters.

In the present work, we discuss further the OPO dynamics for the case of positive not small detuning parameters. We show that in this case the full nondegenerate dynamical equations reduce to a single complex Ginzburg–Landau equation valid near the instability threshold. In an analogy to the superfluid vortices, we introduce the concept of optical vortices in the $\chi^{(2)}$ medium. We give a criterion of supercritical and subcritical instability as well as a condition of the Benjamin–Feir instability.

2. The model equations

In this section we review briefly the OPO dynamics to set the notations. We consider an optical ring cavity with plane mirrors, containing a nonlinear $\chi^{(2)}$ medium (optical parametric oscillator) which converts a field of frequency ω_L into two fields of frequencies ω_S (signal) and ω_I (idler). Three longitudinal modes of the cavity with frequencies ω_0, ω_1 and ω_2 are close to resonance with the field frequencies. In the paraxial approximation the behaviour of the system is governed by a set of coupled dynamical equations [7,10,11]

$$\begin{aligned}
\partial_t A_1 &= \gamma_1[-(1+i\Delta_1)A_1 + ia_1\nabla^2 A_1 + \mu A_2^*] + \gamma_1 A_2^* B, \\
\partial_t A_2 &= \gamma_2[-(1+i\Delta_1)A_2 + ia_2\nabla^2 A_2 + \mu A_1^*] + \gamma_2 A_1^* B, \qquad (2.1)\\
\partial_t B &= \gamma_0[-(1+i\Delta_0)B + ia_0\nabla^2 B] - \gamma_0 A_1 A_2,
\end{aligned}$$

where A_1 and A_2 are the normalized slowly varying envelopes for signal and idler fields, respectively, and $B = A_0 - \mu$, where A_0 is the normalized slowly varying pump field and $\mu = E(1 - i\Delta_0)/(1 + \Delta_0^2)$ is the parametric gain. Here, E is the normalized amplitude of the plain-wave pump input field.

The detuning parameters for pump, signal and idler fields are defined as

$$\Delta_0 = \frac{\omega_0 - \omega_L}{\gamma_0}, \quad \Delta_1 = \frac{\omega_1 - \omega_I}{\gamma_1}, \quad \Delta_2 = \frac{\omega_2 - \omega_S}{\gamma_2},$$

where γ_0, γ_1 and γ_2 are the cavity decay rates of the three fields. ω_0, ω_1 and ω_2 are the three longitudinal cavity frequencies close to the frequencies ω_L, ω_S and ω_I, respectively. The diffraction parameters a_0, a_1 and a_2 for the three fields are defined by

$$a_0 = \frac{c}{2k_z\gamma_0}, \quad a_1 = \frac{c\omega_L}{2\omega_S k_z\gamma_1}, \quad a_2 = \frac{c\omega_L}{2\omega_I k_z\gamma_2},$$

where c is the speed of light and k_z the longitudinal wave vector of the field at frequency ω_L.

3. Ginzburg–Landau equation

In the present work we are interested in the case of positive detunings Δ_1 and Δ_2. According to the stability analysis performed in Ref. [8], the neutral stability curve $\mu = \mu(k)$ has minimum $\mu_c = (1 + \Delta^2)^{1/2}$ at $k_c = 0$. For excitations close to k_c we let $\mu = \mu_c + s\varepsilon^2$, where $s = \pm 1$. The case $s = -1$ corresponds to the subcritical region whereas $s = 1$ to the supercritical case. Following standard multiple-scale expansions, we introduce the new variables, $T_0 = t$, $T_2 = \varepsilon^2 t$, $X = \varepsilon x$ and $Y = \varepsilon y$. We look for solutions for the system (Eq. (2.1)) in the form

$$A_1 = \varepsilon A_1^{(1)} + \varepsilon^3 A_1^{(3)} + O(\varepsilon^5), \tag{3.1}$$

$$A_2 = \varepsilon A_2^{(1)} + \varepsilon^3 A_2^{(3)} + O(\varepsilon^5), \tag{3.2}$$

$$B = \varepsilon^2 B^{(2)} + O(\varepsilon^4), \tag{3.3}$$

Substituting into Eqs. (2.1), we get to first order in ε

$$\begin{aligned} \partial_{T_0} A_1^{(1)} &= -\gamma_1(1 + i\Delta_1)A_1^{(1)} + \gamma_1 \mu_c A_2^{(1)*}, \\ \partial_{T_0} A_2^{(1)} &= -\gamma_2(1 + i\Delta_2)A_2^{(1)} + \gamma_2 \mu_c A_1^{(1)*}, \end{aligned} \tag{3.4}$$

and to order ε^2 we have

$$\partial_{T_0} B^{(2)} = -\gamma_0(1 + i\Delta_0)B^{(2)} - \gamma_0 A_1^{(1)} A_2^{(1)}. \tag{3.5}$$

Finally, to order ε^3 we have

$$\begin{aligned} \partial_{T_0} A_1^{(3)} + \partial_{T_2} A_1^{(1)} = {} & -\gamma_1(1 + i\Delta_1)A_1^{(3)} + i\gamma_1 a_1 \nabla^2 A_1^{(1)} + \gamma_1 \mu_c A_2^{(3)*} \\ & + s\gamma_1 A_2^{(1)*} + \gamma_1 A_2^{(1)*} B^{(2)}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \partial_{T_0} A_2^{(3)} + \partial_{T_2} A_2^{(1)} = {} & -\gamma_2(1 + i\Delta_2)A_2^{(3)} + i\gamma_2 a_2 \nabla^2 A_2^{(1)} + \gamma_2 \mu_c A_1^{(3)*} \\ & + s\gamma_2 A_1^{(1)*} + \gamma_2 A_1^{(1)*} B^{(2)}. \end{aligned} \tag{3.6}$$

We obtain a solution for Eq. (3.4) in the form

$$\begin{aligned} A_1^{(1)}(X, Y; T_2, T_0) &= \mathscr{A}_1(X, Y; T_2)e^{i\omega_c T_0}, \\ A_2^{(1)}(X, Y; T_2, T_0) &= \mathscr{A}_2^*(X, Y; T_2)e^{-i\omega_c T_0}, \end{aligned} \tag{3.8}$$

where

$$\omega_c = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}(\Delta_2 - \Delta_1), \tag{3.9}$$

and

$$\mathscr{A}_2 = \frac{1 + i\Delta}{\mu_c}\mathscr{A}_1. \tag{3.10}$$

Here, $\Delta = (\gamma_1\Delta_1 + \gamma_2\Delta_2)/(\gamma_1 + \gamma_2)$. Since $\mu_c^2 = 1 + \Delta^2$ then Eq. (3.10) yields $|\mathscr{A}_2|^2 = |\mathscr{A}_1|^2$. A possible solution to Eq. (3.5) can be chosen in the form

$$B^{(2)} = -\frac{A_1^{(1)}A_2^{(1)}}{1 + i\Delta_0}. \tag{3.11}$$

In terms of $\mathscr{A}_1$ and $\mathscr{A}_2^*$, the above expression can be rewritten in the form

$$B^{(2)} = -\frac{\mathscr{A}_1\mathscr{A}_2^*}{1 + i\Delta_0} = \sigma|\mathscr{A}_2|^2. \tag{3.12}$$

where $\sigma = -(1 - i\Delta)/(\mu_c + i\mu_c\Delta_0)$. The solvability condition at order ε^3 yields

$$\alpha\partial_{T_2}\mathscr{A}_1 = 2s\mu_c\mathscr{A}_1 + [(a_1 + a_2)\Delta + i(a_1 - a_2)]\nabla^2\mathscr{A}_1 - \frac{4(1 - \Delta\Delta_0)}{1 + \Delta_0^2}|\mathscr{A}_1|^2\mathscr{A}_1, \tag{3.13}$$

where $\alpha = [\gamma_1 + \gamma_2 + i\Delta(\gamma_1 - \gamma_2)]/\gamma_1\gamma_2$. Scaling the spatial variables and time and introducing a new function $\phi = \mathscr{A}_1 \exp[is\Delta(\gamma_1 - \gamma_2)/(\gamma_1 + \gamma_2)]$, we obtain the following equation:

$$\frac{\partial\phi}{\partial t} = s\phi + (1 + i\eta)\nabla^2\phi - \frac{2(1 - \Delta\Delta_0)}{\mu_c(1 + \Delta_0^2)}(1 + iv)|\phi|^2\phi, \tag{3.14}$$

where the parameters η and v are defined as

$$\eta = \frac{(\gamma_1 + \gamma_2)(a_1 - a_2) + \Delta^2(\gamma_2 - \gamma_1)(a_1 + a_2)}{2\Delta(\gamma_1 a_1 + \gamma_2 a_2)},$$

$$v = \frac{\Delta(\gamma_2 - \gamma_1)}{\gamma_1 + \gamma_2}.$$

We see that the full dynamical equations describing the OPO dynamics near the instability threshold for positive detuning parameters are reduced to a complex Ginzburg–Landau equation. Many properties of nonequilibrium systems are encountered in such an equation. It should be noted that this equation provides a quantitative description of real experiments valid only in a small region near the transition threshold.

We notice that in the case where $\Delta\Delta_0 > 1$ there is a subcritical instability of the trivial solution $\phi = 0$, while in the case where $\Delta\Delta_0 < 1$ a supercritical bifurcation takes place.

In the latter case, the Benjamin–Feir instability condition $1 + \eta v < 0$ can be studied. In the corresponding region of parameter space, all spatially periodic solutions of the above Ginzburg–Landau equation are unstable.

Let us note that a complex Ginzburg–Landau equation admits vortex (spiral wave) solutions. Such solutions have the form $\phi(r,\theta) = R(r)e^{i\psi(r)+im\theta}$, where m is the topological charge (usually taken to be ± 1) and (r,θ) are the polar coordinates in two dimensions. The functions $R(r)$ and $\psi(r)$ depend on r only. Analogous to the superfluid vortices the concept of optical vortex was introduced firstly by Coullet et al. in the framework of Maxwell–Bloch equations [12]. This optical vortex corresponds to zeroes of the modulus of the electric field in the transverse plane of the laser beam operating above threshold. Associated to each of these zeroes, a topological charge is defined as the gradient circulation around a closed loop which encloses it. Let us mention also the works of Arecchi et al. [13] where a method for direct detection of topological defects in nonlinear optics was developed.

In conclusion, we have shown that the full OPO dynamical equations for the case of long-wave instability are reduced to a single complex Ginzburg–Landau equation valid near the instability threshold. A criterion for the subcritical as well as for the supercritical instability was given. The Benjamin–Feir instability condition was also established. Finally, for the first time we predict the appearance of optical vortices in a $\chi^{(2)}$ medium.

Acknowledgements

The author is indebted to A.A. Nepomnyashchy and L.M. Pismen for suggesting the problem.

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