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Modulational instability in bulk dispersive quadratically nonlinear media

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Abstract

We analyze stability of continuous wave (cw) solutions to the second-harmonic-generation equations in a multidimensional dispersive medium, and demonstrate that they are always unstable. We also consider the modulational stability in a more general three-wave system, and demonstrate that, at least in some cases, its multidimensional cw solutions may be stable.

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1. Introduction

Solitons in dispersive media with the second-harmonic-generating (SHG) quadratic nonlinearity have recently attracted a great deal of attention, see the papers [1,2,4–8] and references therein. With a few exceptions [4,6], the work was thus far focussed on stationary spatial solitons in one- and two-dimensional media, the latter case assuming cylindrical symmetry [2]. However, the quadratic nonlinearity is expected to produce still more nontrivial results in bulk dispersive media in the form of stable "light bullets", i.e., fully localized spatio-temporal solitons. Unlike the cubic nonlinearity, the quadratic nonlinearity does not give rise to the wave collapse in two- and three-dimensional media, which is why the corresponding "light bullets" have a chance to be stable. The absence of collapse and the possibility of existence of the stable "bullets" in the three-dimensional dispersive SHG model were discovered by Kanashov and Rubenchik back in 1981 [3] (see also [4]). Nevertheless, the spatio-temporal solitons in the two- and three-dimensional SHG models have only recently been constructed [5] (see also [6]), by means of analytical methods based on the variational approximation and parallel direct numerical simulations.

An issue closely related to the existence of both the usual (bright) and dark solitons (in the multidimensional case, the dark solitons represent optical vortices) is the modulational stability of the constant-amplitude background, i.e., cw solutions. For the one-dimensional case, MI has been recently analyzed in [7,8]. It has been demonstrated that, in the usual model of the stationary SHG process in the spatial domain, *all* the cw solutions are modulationally

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unstable. Stable solutions are possible in the temporal domain (e.g., in the model of the SHG optical fiber; however, this is still, chiefly, a theoretical object), provided that the signs of the dispersion coefficients at the fundamental harmonic (FH) and second harmonic (SH) have opposite signs [8].

The main objective of the present work is to study MI of the cw solutions in the multidimensional dispersive SHG model. We show that, in the multidimensional case, cw solutions are always unstable. This includes the case in which the FH and SH dispersion coefficients have opposite signs, even though in this case the modulational stability is possible in one dimension, as it was demonstrated in [8]. In particular, this implies that optical vortices with a modulationally stable cw background can *never* exist in the SHG media, unlike stable multidimensional bright solitons.

The general equations describing copropagation of FH and SH in the quadratic medium were actually formulated by Kanashov and Rubenchik in 1981 [3]:

$$iu_z + \nabla_\perp^2 u + u_{\tau\tau} - u + vu^* = 0, \tag{1.1}$$

$$2iv_z + \nabla_\perp^2 v + \delta v_{\tau\tau} - \gamma v + \tfrac{1}{2}u^2 = 0, \tag{1.2}$$

where u and v are slowly varying envelopes of FH and SH, z and τ are the properly rescaled propagation distance and the so-called local time, the d-dimensional gradient $\nabla_\perp$ acts upon the transverse coordinates ($d = 1$ and $d = 2$, respectively, for the two- and three-dimensional cases), and the parameter γ measures the phase mismatch between the two harmonics. The dispersion at the FH is assumed to be anomalous, and δ is the ratio of the SH and FH dispersion coefficients, which may have any sign. The system (1.1) and (1.2) implicitly assumes equal group velocities at the chosen carrier wavelengths, which can always be achieved by means of the necessary gauge transformation.

2. Modulational instability of the homogeneous state

First of all, we will study MI of the homogeneous solution to the system (1.1) and (1.2). This solution is given by $u_0 = \sqrt{2\gamma}$ and $v_0 = 1$, where we assume $\gamma > 0$, and a constant phase of the solution can be removed in an obvious way. We study stability of this solution by adding to it an infinitesimal perturbation, $u = u_0 + u_1$ and $v = v_0 + v_1$. Then Eqs. (1.1) and (1.2) yield

$$iu_{1z} + \nabla_\perp^2 u_1 + u_{1\tau\tau} - u_1 + u_1^* + u_0 v_1 = 0, \tag{2.1}$$

$$2iv_{1z} + \nabla_\perp^2 v_1 + \delta v_{1\tau\tau} - \gamma v_1 + u_0 u_1 = 0. \tag{2.2}$$

Next we look for a solution to the system (2.1) and (2.2) with each perturbation being a linear combination of $\exp(i(\mathbf{k}\cdot\mathbf{x}_\perp + \omega\tau) + \sigma z)$ and $\exp(-i(\mathbf{k}\cdot\mathbf{x}_\perp + \omega\tau) + \sigma^* z)$, where σ is the (generally complex) instability growth rate, and ω and $\mathbf{k}$ are the frequency and wave vector for the perturbation. It follows that the stability determinant of the perturbed cw solution is

$$\begin{pmatrix} i\sigma - B & 1 & \sqrt{2\gamma} & 0 \\ 1 & -i\sigma - B & 0 & \sqrt{2\gamma} \\ \sqrt{2\gamma} & 0 & 2i\sigma - A & 0 \\ 0 & \sqrt{2\gamma} & 0 & -2i\sigma - A \end{pmatrix}, \tag{2.3}$$

where $A \equiv k^2 + \delta\omega^2 + \gamma$ and $B \equiv k^2 + \omega^2 + 1$. The equation for the instability eigenvalue σ produced by this determinant is

$$4\sigma^4 + \sigma^2[A^2 + 4B^2 - 4 + 8\gamma] + A^2(B^2 - 1) - 4\gamma BA + 4\gamma^2 = 0. \tag{2.4}$$

The solution to Eq. (2.4) for σ^2 is

$$8\sigma^2 = -[A^2 + 4(B^2 - 1) + 8\gamma] \pm \sqrt{\mathcal{D}}, \tag{2.5}$$

where

$$\mathcal{D} \equiv [A^2 - 4(B^2 - 1)]^2 + 16\gamma[(A + 2B)^2 - 4]. \tag{2.6}$$

Because we are dealing with a conservative (Hamiltonian) system, the stability may be only neutral. Therefore σ must be purely imaginary, i.e., σ^2 must be strictly real and nonpositive, for the stability. Any complex or real but positive σ^2 implies an instability.

Although we have the explicit formula (2.5) for σ, the full analysis is not straightforward. Therefore, we will study the system in different regimes. First, we consider the homogeneous perturbations, i.e., $k = \omega = 0$. In this case, $A = \gamma$ and $B = 1$, which yields $\sigma^2 = 0$ or $\sigma^2 = -\gamma(\gamma + 8)/4$. Since $\gamma > 0$, the stability condition is satisfied for this perturbation.

Next, we consider the long-wave approximation, in which we assume that ω , k and σ are of the same order of smallness. Then Eq. (2.4) becomes

$$\sigma^2(\gamma^2 + 8\gamma) + 2\gamma^2(k^2 + \omega^2) - 4\gamma[\gamma(k^2 + \omega^2) + (k^2 + \delta\omega^2)] = 0. \tag{2.7}$$

Solving for σ, we obtain

$$\sigma^2 = \frac{(2\gamma + 4)k^2 + (2\gamma + 4\delta)\omega^2}{\gamma + 8}. \tag{2.8}$$

Direct inspection of this expression demonstrates that the parameter σ^2 cannot be made negative definite. This means that the homogeneous cw solution cannot be stable and, therefore, it cannot support a stable optical vortex either. On the other hand, the homogeneous solutions are stable against short-wave perturbations as can be seen directly from Eq. (2.5). Indeed, for large k and ω one obtains, for the two branches of expression (2.5), $\sigma_+^2 \approx -4B^2$ and $\sigma_-^2 \approx -2A^2$.

In the case when the cw solutions are unstable, it is interesting to find a particular perturbation mode that provides for a maximum instability growth rate. In the generic case, such a perturbation mode will determine periodicity of a nontrivial solution generated by MI [9]. To this end, in Fig. 1 we display a three-dimensional plot of the instability growth rate as a function of the perturbation parameters k and ω, as obtained numerically from the full equation (2.5).

Another nontrivial solution to the system (1.1) and (1.2) can be obtained by means of the transformation $u = U\mathrm{e}^{-2\mathrm{i}z}, v = -V\mathrm{e}^{-4\mathrm{i}z}$ (which is *not* equivalent to the Galileian transformation). The transformed variables U and V satisfy the following equations:

$$\mathrm{i}U_z + \nabla_\perp^2 U + U_{\tau\tau} + U - VU^* = 0, \tag{2.9}$$

$$2\mathrm{i}V_z + \nabla_\perp^2 V + \delta V_{\tau\tau} + \gamma_s V - \tfrac{1}{2}U^2 = 0, \tag{2.10}$$

where $\gamma_s \equiv 8 - \gamma$. Following the same analysis, one finds that the solution $U_0 = \sqrt{2\gamma_s}$, $V_0 = 1$ is still stable against the homogeneous perturbation. Moreover, in the long-wave approximation, we have

$$\sigma^2 = -\frac{(2\gamma_s + 4)k^2 + (2\gamma_s + 4\delta)\omega^2}{\gamma_s + 8}. \tag{2.11}$$

One can see from Eq. (2.11) that the homogeneous solution to Eqs. (2.9) and (2.10) is stable against long-wave perturbations. However, it proves to be unstable against other infinitesimal perturbations. In Fig. 2 we show a three-dimensional plot of the instability growth rate $\mathrm{Re}(\sigma)$ as a function of the perturbation parameters k and ω.

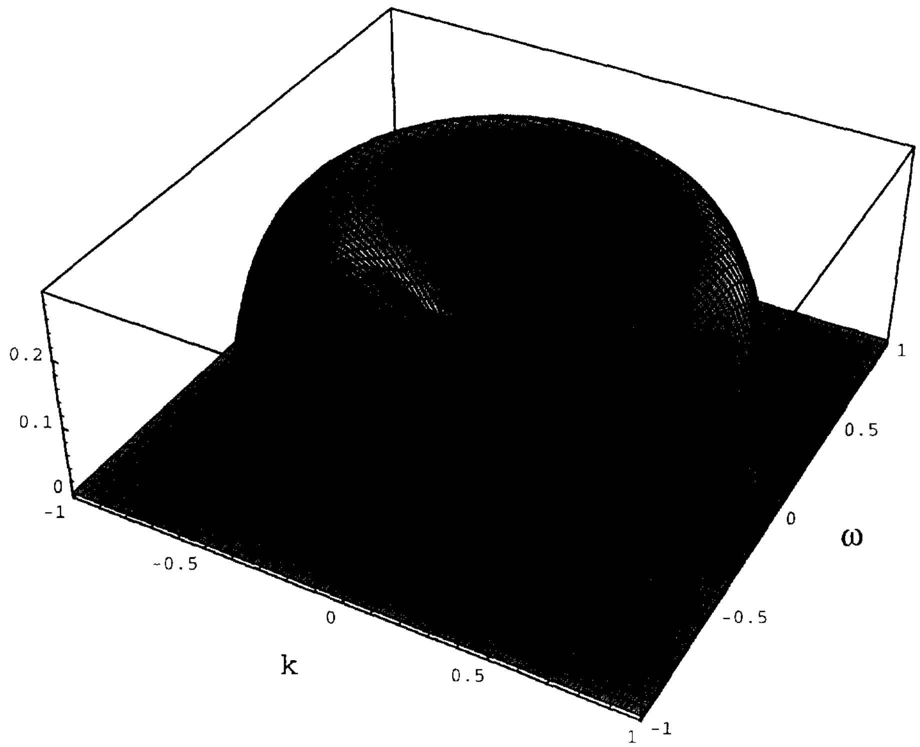

Fig. 1. A three-dimensional plot of the instability growth rate $\mathrm{Re}(\sigma)$ (see Eq. (2.5)) for the homogeneous solutions to Eqs. (2.1) and (2.2) as a function of the perturbation parameters k and ω for $\gamma = 0.5$ and $\delta = 1$.

In accord with Eq. (2.11), for k and ω close to the origin there is no instability, while a strong instability occurs at finite k and ω. Increasing the parameter γ_s, the instability region shrinks, so that for a large mismatch $\gamma_s \approx 20$, the solution becomes almost stable for all values of k. This is illustrated by a contour plot of the function $\mathrm{Re}\,\sigma(k, \omega)$ at $\gamma_s = 20$ and $\delta = 1$ (Fig. 3). One way to understand this result is to consider the system (2.9) and (2.10) in the limit $\gamma_s \to \infty$. To the first order, this system is equivalent to a defocusing nonlinear Schrödinger equation, in which the homogeneous solution is well-known to be modulationally stable.

3. Modulational instability for the general cw solutions

In this section we will analyze MI for a general cw solution to Eqs. (1.1) and (1.2) in the form

$$u_0 = A_0 e^{i\omega_0\tau}, \qquad v_0 = B_0 e^{2i\omega_0\tau}, \tag{3.1}$$

where $B_0 = 1+\omega_0^2$ and $A_0 = \sqrt{2(1+\omega_0^2)(\gamma + 4\delta\omega_0^2)}$. More general cw solutions contain the additional phase terms $\mathbf{k}_0 \cdot \mathbf{x}_\perp$ and $q_0 z$. However, since Eqs. (1.1) and (1.2) are Galileian invariant, $\mathbf{k}_0$ and q_0 can be transformed to zero.

To proceed with the stability analysis, we again let $u = u_0 + u_1 e^{i\omega_0\tau}$ and $v = v_0 + v_1 e^{2i\omega_0\tau}$. Substituting this into Eqs. (1.1) and (1.2), linearizing around the cw solution, we obtain

$$i\partial_z u_1 + \nabla_\perp^2 u_1 + (\partial_\tau + i\omega_0)^2 u_1 - u_1 + B_0 u_1^* + A_0 v_1 = 0, \tag{3.2}$$

$$2i\partial_z v_1 + \nabla_\perp^2 v_1 + \delta(\partial_\tau + 2i\omega_0)^2 v_1 - \gamma v_1 + A_0 u_1 = 0. \tag{3.3}$$

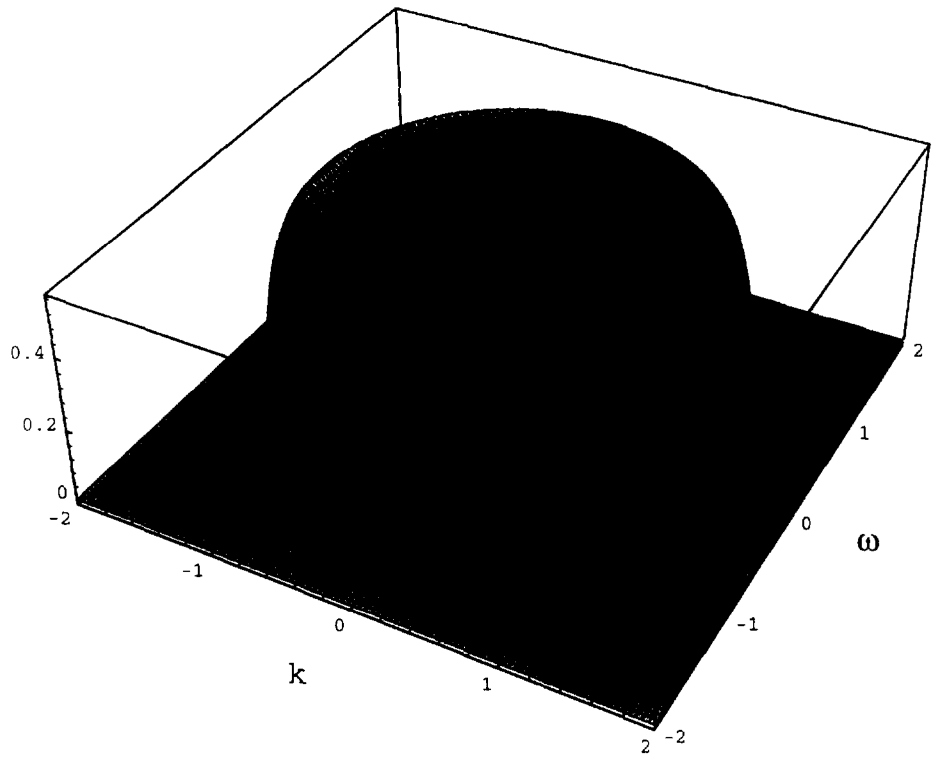

Fig. 2. A three-dimensional plot of the instability growth rate Re(σ) for the homogeneous solution to Eqs. (2.9) and (2.10) as a function of the perturbation parameters k and ω for $\gamma_s = 0.5$ and $\delta = 1$.

Note that, in the particular case of the τ-independent solutions, Eqs. (3.2) and (3.3) take the form of the stability problem for a one-dimensional system. However, it is easy to check that these equations are really tantamount to the previously studied one-dimensional case [7,8] only if $\omega = 0$; otherwise, the values of the coefficients are different, hence the stability problem should be studied anew.

We look for a solution in essentially the same form as in the previous section, i.e., as a superposition of $\exp(\mathrm{i}(\mathbf{k}\cdot\mathbf{x}_\perp + \omega\tau) + \sigma z)$ and $\exp(-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}_\perp + \omega\tau) + \sigma^* z)$. Substitution of these wave forms into the linearized equations yields the following determinant of the linearized problem:

$$\begin{pmatrix} \mathrm{i}\sigma - b_1 & B_0 & A_0 & 0 \\ B_0 & -\mathrm{i}\sigma - b_2 & 0 & A_0 \\ A_0 & 0 & 2\mathrm{i}\sigma - a_1 & 0 \\ 0 & A_0 & 0 & -2\mathrm{i}\sigma - a_2 \end{pmatrix}, \tag{3.4}$$

where

$$a_1 \equiv k^2 + \delta(\omega + 2\omega_0)^2 + \gamma, \qquad a_2 \equiv k^2 + \delta(\omega - 2\omega_0)^2 + \gamma,$$
$$b_1 \equiv k^2 + (\omega + \omega_0)^2 + 1, \qquad b_2 \equiv k^2 + (\omega - \omega_0)^2 + 1,$$

hence the stability eigenvalues are determined by the algebraic equation

$$(2\mathrm{i}\sigma - a_1)(-2\mathrm{i}\sigma - a_2)[(\mathrm{i}\sigma - b_1)(-\mathrm{i}\sigma - b_2) - B_0^2] - A_0^2[(2\mathrm{i}\sigma - a_1)(\mathrm{i}\sigma - b_1)$$
$$+(-\mathrm{i}\sigma - b_2)(-2\mathrm{i}\sigma - a_2)] + A_0^4 = 0. \tag{3.5}$$

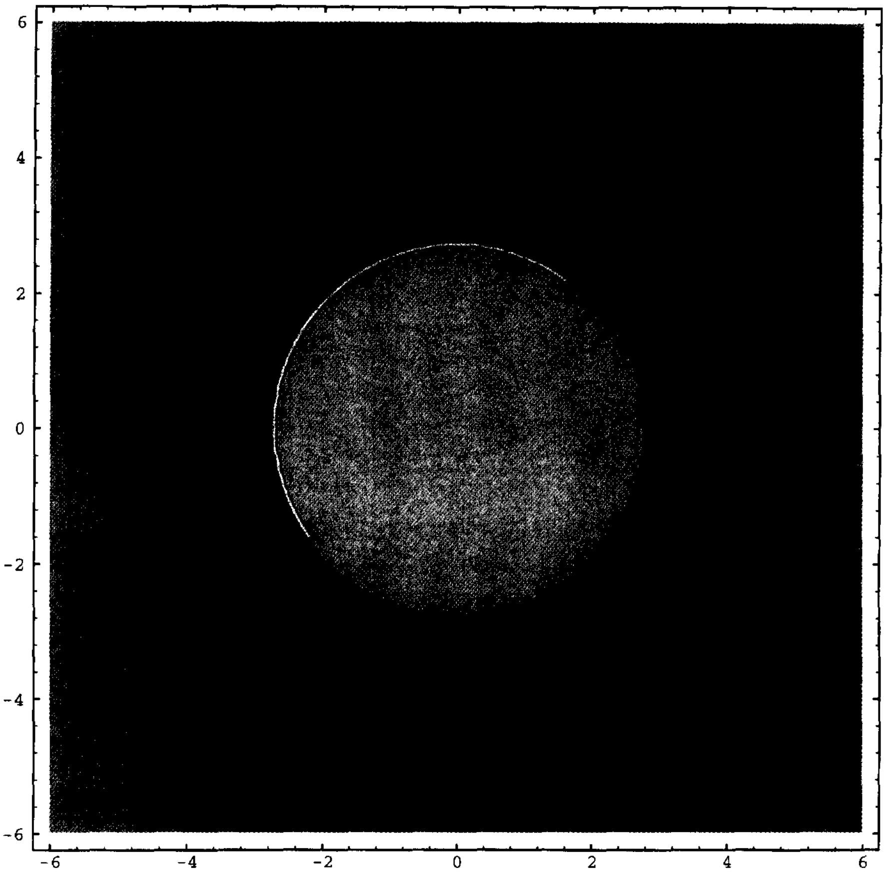

Fig. 3. A contour plot of the instability growth rate $\mathrm{Re}(\sigma)$ for $\gamma_s = 20$ and $\delta = 1$.

First we consider the homogeneous perturbation, $\omega = k = 0$. In this case we have $a_1 = a_2 = \gamma + 4\delta\omega_0^2$ and $b_1 = b_2 = 1 + \omega_0^2$. Substituting this into Eq. (3.5), we find

$$\sigma^2 = -\frac{(\gamma + 4\delta\omega_0^2)(4\delta\omega_0^2 + 8\omega_0^2 + \gamma + 8)}{4}. \tag{3.6}$$

To find the stability region, we require, as before, $\sigma^2 \le 0$. This yields

$$\gamma \ge -4\delta\omega_0^2. \tag{3.7}$$

This result is consistent with that obtained in the previous section for the case $\omega_0 = 0$.

In the case when k, ω and σ are finite but small (the long-wave approximation), Eq. (3.5) can be written in the form

$$[(\gamma + 4\delta\omega_0^2)^2 + 4A_0^2]\sigma^2 + 4\mathrm{i}\omega_0(\gamma + 4\delta\omega_0^2)\eta\omega\sigma - 2(\gamma + 4\delta\omega_0^2)[\alpha k^2 + \beta\omega^2] = 0, \tag{3.8}$$

where

$$\eta = 4(1 + \omega_0^2)(1 + \delta) + 4\delta\omega_0^2 + \gamma,$$
$$\alpha = (1 + \omega_0^2)(4\delta\omega_0^2 + 2\omega_0^2 + \gamma + 2),$$

$$\beta = 30\delta\omega_0^4 + 24\delta\omega_0^2 + 3\gamma\omega_0^2 + \gamma + 2\delta.$$

The neutral stability (purely imaginary σ) can be achieved by requiring $\Delta < 0$, where

$$\Delta \equiv -8(\gamma + 4\delta\omega_0^2)^2[2\omega_0^2\eta^2 - (4\delta\omega_0^2 + 8\omega_0^2 + \gamma + 8)\beta]\omega^2 \\ +8(\gamma + 4\delta\omega_0^2)^2(4\delta\omega_0^2 + 8\omega_0^2 + \gamma + 8)\alpha k^2. \tag{3.9}$$

Under condition (3.7), one notices that Δ is positive, hence the general cw solution is *always unstable* against the long-wave perturbations.

4. Modulational stability of three-wave interactions in the dispersive medium

In this section we aim to analyze MI in a dispersive optical media supporting more general three-wave (3W) nonlinear quadratic interactions. According to [10], the model equations are

$$\mathrm{i}(E_p)_z + \tfrac{1}{2}D_p(E_p)_{\tau\tau} = E_1E_2, \tag{4.1}$$

$$\mathrm{i}(E_1)_z + \mathrm{i}\nu_1(E_1)_\tau + \tfrac{1}{2}D_1(E_1)_{\tau\tau} = E_pE_2^*, \tag{4.2}$$

$$\mathrm{i}(E_2)_z + \mathrm{i}\nu_2(E_2)_\tau + \tfrac{1}{2}D_2(E_2)_{\tau\tau} = E_pE_1^*, \tag{4.3}$$

where E_p and E_n ($n = 1, 2$) represent the envelopes of the pump wave (PW) and two daughter waves (DW), respectively, D_p and D_n are the corresponding dispersion coefficients, and the coefficient of the parametric interaction is set equal to one. The coefficients ν_n measure the group velocities of the DWs in the reference frame in which the PW is at rest. The remaining invariance of the dispersive 3W equations with respect to the gauge transformations is employed to set the phase-velocity coefficients equal to zero in all the three equations. We consider a cw solution for system (4.1)–(4.3) in the form

$$E_p^0 = A_p\mathrm{e}^{\mathrm{i}k_pz}, \qquad E_n^0 = A_n\mathrm{e}^{\mathrm{i}k_nz}, \tag{4.4}$$

with $k_p = k_1 + k_2$, $|A_p|^2 = k_1k_2$, $|A_1|^2 = k_2(k_1 + k_2)$, and $|A_2|^2 = k_1(k_1 + k_2)$. We linearize the system around this cw solution by writing

$$E_p = E_p^0 + \mathcal{E}_p\mathrm{e}^{\mathrm{i}k_pz}, \tag{4.5}$$

$$E_1 = E_1^0 + \mathcal{E}_1\mathrm{e}^{\mathrm{i}k_1z}, \tag{4.6}$$

$$E_2 = E_2^0 + \mathcal{E}_2\mathrm{e}^{\mathrm{i}k_2z}. \tag{4.7}$$

Substituting this into Eqs. (4.1)–(4.3), one obtains

$$(\mathrm{i}\partial_z - k_p)\mathcal{E}_p + \tfrac{1}{2}D_p(\mathcal{E}_p)_{\tau\tau} - A_1\mathcal{E}_2 - A_2\mathcal{E}_1 = 0, \tag{4.8}$$

$$(\mathrm{i}\partial_z - k_1)\mathcal{E}_1 + \mathrm{i}\nu_1(\mathcal{E}_1)_\tau + \tfrac{1}{2}D_1(\mathcal{E}_1)_{\tau\tau} - A_p\mathcal{E}_2^* - A_2^*\mathcal{E}_p = 0, \tag{4.9}$$

$$(\mathrm{i}\partial_z - k_2)\mathcal{E}_2 + \mathrm{i}\nu_2(\mathcal{E}_2)_\tau + \tfrac{1}{2}D_2(\mathcal{E}_2)_{\tau\tau} - A_p\mathcal{E}_1^* - A_1^*\mathcal{E}_p = 0. \tag{4.10}$$

Next we assume, as in the previous sections, that each perturbation is a linear combination of $\exp(\mathrm{i}\omega\tau + \lambda z)$ and $\exp(-\mathrm{i}\omega\tau + \lambda^* z)$, where λ is the instability growth rate, and ω is an arbitrary frequency of the small perturbation. The determinant of the linearized equations for the perturbed cw solutions is

$$\begin{pmatrix} z_1 & 0 & -A_1 & 0 & -A_2 & 0 \\ 0 & z_2 & 0 & -A_1^* & 0 & -A_2^* \\ -A_2^* & 0 & 0 & -A_p & z_3 & 0 \\ 0 & -A_2 & -A_p^* & 0 & 0 & z_4 \\ -A_1^* & 0 & z_5 & 0 & 0 & -A_p \\ 0 & -A_1 & 0 & z_6 & -A_p^* & 0 \end{pmatrix}, \tag{4.11}$$

where

$$z_1 = \mathrm{i}\lambda - k_p - \tfrac{1}{2}D_p\omega^2, \qquad z_2 = z_1^*,$$
$$z_3 = \mathrm{i}\lambda - k_1 - \nu_1\omega - \tfrac{1}{2}D_1\omega^2, \qquad z_4 = -\mathrm{i}\lambda - k_1 + \nu_1\omega - \tfrac{1}{2}D_1\omega^2,$$
$$z_5 = \mathrm{i}\lambda - k_2 - \nu_2\omega - \tfrac{1}{2}D_2\omega^2, \qquad z_6 = -\mathrm{i}\lambda - k_2 + \nu_2\omega - \tfrac{1}{2}D_2\omega^2.$$

The stability condition requires that all the real parts of the eigenvalues of the above determinant be negative or exactly zero. However, finding the eigenvalues of (4.11) is a formidable task. Hence we start the analysis with the case of the uniform perturbation, $\omega = 0$. In this case, the growth rate λ is given by

$$\lambda^2 = -4(k_1^2 + k_1k_2 + k_2^2). \tag{4.12}$$

The quadratic form on the right-hand side of this expression is negative definite, hence the solution is always stable against the uniform perturbations.

Because it is too difficult to consider stability against nonuniform perturbations in the general case, in what follows we assume that λ and ω are small, and retain terms of the fourth order in λ and ω (the terms of the second order are identically zero). After lengthy calculations we find that the instability growth rate λ satisfies an algebraic equation,

$$c_0\xi^4 + c_1\xi^3 - c_2\xi^2 + c_3\xi + c_4 = 0, \tag{4.13}$$

where $\lambda \equiv \mathrm{i}\omega\xi$, and

$$c_0 = 4(k_1^2 + k_1k_2 + k_2^2),$$
$$c_1 = 2\nu_1[(k_2 + k_p)^2 + k_1^2] + 2\nu_2[(k_1 + k_p)^2 + k_2^2],$$
$$c_2 = 2k_1k_2k_p(D_1 + D_2 - D_p) - \nu_1^2(k_2 + k_p)^2 - \nu_2^2(k_1 + k_p)^2 - 2\nu_1\nu_2(4k_p^2 - k_1k_2),$$
$$c_3 = 2k_p\nu_1\nu_2[\nu_1(k_2 + k_p) + \nu_2(k_1 + k_p)] - 2k_1k_2k_p[\nu_1(D_2 - D_p) + \nu_2(D_1 - D_p)],$$
$$c_4 = -k_1k_2k_p[D_1D_pk_2 + D_2D_pk_1 + D_1D_2k_p] + \nu_1\nu_2k_p(2k_1k_2D_p + k_p\nu_1\nu_2)\,.$$

The stability of the solution (4.4) requires ξ to be real. Let us first consider the special case when the phase-mismatch coefficients in the underlying equations (4.1)–(4.3) are zero, $\nu_1 = \nu_2 = 0$. In this case, $c_1 = c_3 \equiv 0$, leaving Eq. (4.13) in the form

$$c_0\xi^4 - c_2\xi^2 + c_4 = 0. \tag{4.14}$$

If c_2 and c_4 are strictly positive and $c_2^2 - 4c_0c_4 > 0$, it follows from Eq. (4.14) that the solution (4.4) is stable. It should be pointed out that these conditions do not hold for every k_1 and k_2. In particular, if $k_1 = k_2 = k > 0$ (< 0), the stability condition takes the form of the following system of inequalities:

$$D_1 + D_2 - D_p > 0\ (< 0), \qquad D_p(D_1 + D_2) + 2D_1D_2 < 0,$$
$$(D_1 + D_2 - D_p)^2 + 6(D_pD_1 + D_pD_2 + 2D_1D_2) > 0.$$

For nonzero mismatch parameters ν_1 and ν_2, a simple result can be obtained in the case $\nu_1 = -\nu_2 \equiv \nu$. In addition, we will assume that $k_1 = k_2 = k$ and $D_1 = D_2 = -D_p \equiv D$, then $c_1 = c_3 = 0$. In this case, the final stability condition for solutions with $kD > 0$ is $\nu^2 < 3kD$.

5. Conclusion

We have analyzed stability of the homogeneous as well as general cw solution for the second-harmonic-generation equations describing copropagation of the fundamental and second harmonics in the quadratically nonlinear medium. We have shown that the cw solutions are always unstable, against long-wave perturbation for one class of solutions, and against short-wave perturbations for another class. From here, we conclude that stable optical vortices cannot be supported by the second-harmonic-generating media.

On the other hand, we have found that a general three-wave system with the quadratic nonlinearities may have, at least in some special cases, cw solutions stable against both long- and short-wave perturbations, i.e., it has a potential to support stable vortices. The vortices will be considered in a separate work.

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