Numerical Functional Analysis and Optimization, 30(1–2):46–69, 2009

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560802679398

MODULATION THEORY FOR SELF-FOCUSING IN THE NONLINEAR SCHRÖDINGER–HELMHOLTZ EQUATION

Yanping Cao[1], Ziad H. Musslimani[2], and Edriss S. Titi[3]

[1]*Department of Mathematics, University of California, Irvine, California, USA*
[2]*Department of Mathematics, Florida State University, Tallahassee, Florida, USA*
[3]*Department of Mathematics, and Department of Mechanical and Aerospace Engineering, University of California, Irvine, California, USA and Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot, Israel*

□ *The nonlinear Schrödinger–Helmholtz (SH) equation in N space dimensions with 2σ nonlinear power was proposed as a regularization of the classic nonlinear Schrödinger (NLS) equation. It was shown that the SH equation has a larger regime $\left(1 \leq \sigma < \frac{4}{N}\right)$ of global existence and uniqueness of solutions compared with that of the classic NLS $\left(0 < \sigma < \frac{2}{N}\right)$. In the limiting case where the Schrödinger–Helmholtz equation is viewed as a perturbed system of the classic NLS equation, we apply modulation theory to the classic critical case ($\sigma = 1, N = 2$) and show that the regularization prevents the formation of singularities of the NLS equation. Our theoretical results are supported by numerical simulations.*

Keywords Hamiltonian; Modulation theory; Perturbed critical nonlinear Schrödinger equation; Regularization of the nonlinear Schrödinger equation; Schrödinger–Helmholtz equation; Schrödinger–Newton equation.

AMS Subject Classification 35Q40; 35Q55; 78A60.

1. INTRODUCTION

The nonlinear Schrödinger–Helmholtz (SH) system is given as below

$$\begin{aligned} &iv_t + \triangle v + u|v|^{\sigma-1}v = 0, \quad t > 0, \ x \in \mathbb{R}^N, \\ &u - \alpha^2 \triangle u = |v|^{\sigma+1}, \\ &v(0) = v_0 \end{aligned} \tag{1.1}$$

Address correspondence to Edriss S. Titi, Department of Mathematics, University of California, Irvine, CA 92697-3875, USA; E-mail: etiti@math.uci.edu

where $\sigma \geq 1$ and, for simplicity, $\alpha > 0$. This system has been proposed [5] as a regularization for the classic nonlinear Schrödinger (NLS) equation:

$$\begin{aligned} &iu_t + \triangle u + |u|^{2\sigma} u = 0, \quad t > 0, \ \ x \in \mathbb{R}^N, \\ &u(0) = u_0. \end{aligned} \tag{1.2}$$

In [5] we showed global existence of solution of the Cauchy problem (1.1) for $1 \leq \sigma < 3$ when $N = 1$ and $1 \leq \sigma < \frac{4}{N}$ when $N > 1$. It is well known that the classic NLS has global solution for $0 \leq \sigma < \frac{2}{N}$ in any dimension $N \geq 1$ and there is finite time blow up in the critical case $\sigma = \frac{2}{N}$(see, e.g., [6, 13–15, 19, 20] and references therein). So we regard the SH system (1.1) as a regularization system for the NLS (1.2) as the former system has larger regime of global existence for the parameter σ, which contains the values for which the NLS (1.2) blows up. Note that (1.1) is a Hamiltonian system with the corresponding Hamiltonian

$$\mathscr{H}(v) = \int_{\mathbb{R}^N} \left(|\nabla v(x,t)|^2 - \frac{u(x,t)|v(x,t)|^{\sigma+1}}{\sigma+1} \right) dx$$

and can be obtained formally by the variational principle

$$i\frac{\partial v}{\partial t} = \frac{\delta \mathscr{H}(v)}{\delta v^*},$$

where v^* denotes the complex conjugate of v. Let us rewrite system (1.1) as

$$iv_t + \triangle v + |v|^{2\sigma} v + \alpha^2 (\triangle u)|v|^{\sigma-1} v = 0, \tag{1.3}$$

where $u = (I - \alpha^2 \triangle)^{-1}(|v|^{\sigma+1})$. Observe that when the parameter α goes to zero, one can regard system (1.1) (or 1.3) as a formal perturbation of the classic NLS. There has been a lot of work on perturbed NLS in the critical case $\sigma = \frac{2}{N}$ (see, e.g., [8–10], and references therein). In this paper, we will apply modulation theory (see, e.g., [11, 12, 17, 18] for references about modulation theory) to the classic critical case $\sigma = 1, N = 2$ in order to shed more light on the nature of the effect of the regularization in preventing the blow up. In this case, the classic NLS blows up for certain initial data, however, the SH system has global solution with the regularization parameter $\alpha > 0$. Indeed, modulation theory tries to explain the role of the regularization in preventing the formation of singularity near the critical values of the initial data that blow up in the classic case. Intuitively speaking, the basic idea behind modulation theory is that the energy near singularity is equal to the power of the Townes soliton [see Eq. (2.1) below], and the profiles of the solutions are asymptotic to some rescaled profiles of the Townes soliton. With modulation theory, one can reduce the perturbed system (1.3) into a simpler system of ordinary differential equations that do not depend on the spatial variables, and they are supposed to be easier to analyze both analytically and numerically.

In this paper, we will study how the parameter α prevents the singularity formation in the 2D critical NLS. The work here will follow the study we initiated in [5]. In particular, some of the statements have already been mentioned there. For the sake of completeness, we will restate some important theorems and propositions.

2. MODULATION THEORY

First we review some main results on modulation theory for the unperturbed critical NLS following [12]. As stated in [12], most of the results presented in this section are formal and have not been made rigorous at present. We emphasize here that we consider the case $\sigma = 1, N = 2$ in order to see how the regularization prevents singularity formation.

In the case of self-focusing, the amount of power that goes into the singularity is equal to the critical power $N_c = \|R\|_2^2 = \int_0^\infty R^2(r) r\, dr$ where R, the Townes soliton, is the solution of the following equation with minimal L^2-norm, which is positive and radially symmetric

$$\triangle R - R + R^3 = 0, \quad R'(0) = 0, \quad \lim_{r \to +\infty} R(r) = 0, \quad \triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{2.1}$$

Close to the stage of self-focusing, the solution of (1.2) separates into two components as it propagates,

$$v = v_s + v_{\text{back}} \tag{2.2}$$

where v_s is the high-intensity inner core of the beam that self-focuses toward its center axis and v_{back} is the low-intensity outer part that propagates forward following the usual linear propagation mode. Close enough to the singularity, v_s has only small excess power above the critical one and approches the radially symmetric asymptotic profile:

$$v_s(x,t) = \frac{1}{L(t)} V(\tau, \xi) \exp\left(i \left(\tau + \frac{L_t}{L}\frac{r^2}{4} \right) \right), \quad arg\, V(\tau, 0) = 0, \tag{2.3}$$

where $L(t)$ is a yet undetermined function that is used to rescale the variables $\xi = (\xi_1, \xi_2), x = (x_1, x_2)$ with

$$\xi = \frac{x}{L}, \quad \frac{d\tau}{dt} = \frac{1}{L^2}.$$

Then the reduced system for unperturbed critical NLS is

$$L_{tt} = -\frac{\beta}{L^3}, \qquad \beta_t = -\frac{e^{-\frac{\pi}{\sqrt{\beta}}}}{L^2}, \tag{2.4}$$

where $L(t)$ is the scaling factor and β is proportional to the excess of the power near singularity: $\beta = M(N - N_c)$ for constant $M = \frac{1}{4}\int_0^\infty R^2(\rho)\rho^3 d\rho \approx 0.55$. We emphasize here that in the case of self-focusing of the original system, both the scaling factor L and the excess energy β will approach zero.

Next we review some results on modulation theory for the perturbed critical NSL [12]. For a general perturbed critical NLS of the form

$$iv_t + \triangle v + |v|^2 v + \epsilon F(v, v_t, \nabla v, \dots) = 0, \quad |\epsilon| \ll 1, \tag{2.5}$$

where F is an even function in x, modulation theory is valid when the following three conditions hold.

Condition 2.1. The focusing part of the solution is close to the asymptotic profile

$$v_s(t, x) \sim \frac{1}{L(t)} V(\tau, \xi) \exp\left[i\tau(t) + i\frac{L_t}{L}\frac{r^2}{4}\right], \tag{2.6}$$

where

$$\xi = \frac{x}{L}, \quad r^2 = x_1^2 + x_2^2, \quad \frac{d\tau}{dt} = \frac{1}{L(t)^2}$$

and $V = R + \mathcal{O}(\beta, \epsilon)$, $\beta = -L^3 L_{tt}$ and R is the Townes soliton given in (2.1).

Condition 2.2. The power is close to critical

$$\left| \frac{1}{2\pi} \int |v_s(t, x_1, x_2)|^2 dx_1\, dx_2 - N_c \right| \ll 1, \tag{2.7}$$

or, equivalently,

$$|\beta(t)| \ll 1, \tag{2.8}$$

where $N_c = \frac{1}{2\pi}\int_{\mathbb{R}^2} R(x_1, x_2)^2 dx_1\, dx_2 = \|R\|_2^2$ is the threshold energy of blow up.

Condition 2.3. The perturbation ϵF is small in comparison with the other terms, i.e.,

$$|\epsilon F| \ll |\triangle v|, \quad |\epsilon F| \ll |v|^3. \tag{2.9}$$

The following proposition is given in [12].

Proposition 2.4. *If Conditions* 2.1–2.3 *hold, self-focusing in the perturbed critical NLS* (2.5) *is given to leading order by the reduced system*

$$\beta_t + \frac{e^{-\frac{\pi}{\sqrt{\beta}}}}{L^2} = \frac{\epsilon}{2M}(f_1)_t - \frac{2\epsilon}{M}f_2, \quad L_{tt} = -\frac{\beta}{L^3}. \tag{2.10}$$

The auxiliary functions f_1, f_2 *are given by*

$$f_1(t) = 2L(t)Re\left[\frac{1}{2\pi}\int_{\mathbb{R}^2} F(\psi_R)\exp(-iS)[R(\rho) + \rho\nabla R(\rho)]dx_1\, dx_2\right], \tag{2.11}$$

$$f_2(t) = Im\left[\frac{1}{2\pi}\int_{\mathbb{R}^2} \psi_R^* F(\psi_R)dx_1\, dx_2\right], \tag{2.12}$$

where

$$\psi_R = \frac{1}{L}R(\rho)\exp(iS),$$

and R *is the Townes soliton given in* (2.1), $\rho = \frac{r}{L}$, $S = \tau(t) + \frac{L_t}{L}\frac{r^2}{4}$, $\frac{d\tau}{dt} = \frac{1}{L^2}$, $M = \frac{1}{4}\int_0^\infty R(\rho)^2\rho^3 d\rho \approx 0.55$.

Furthermore, if F *is a conservative perturbation, i.e.,*

$$Im\int_{\mathbb{R}^2} v^* F(v)dx_1\, dx_2 = 0, \tag{2.13}$$

then $f_2 = 0$. *Because* $\beta \ll 1$, $e^{-\frac{\pi}{\sqrt{\beta}}} \ll \beta$, *taking the leading order by neglecting the exponential term in the first equation of* (2.10), *we further reduce the system* (2.10) *into the following system*

$$-L^3 L_{tt} = \beta = \beta_0 + \frac{\epsilon}{2M}f_1, \quad \beta_0 = \beta(0) - \frac{\epsilon}{2M}f_1(0), \tag{2.14}$$

where β_0 *is independent of* t.

In general, at the onset of self-focusing only Condition 2.3 holds. Therefore, if the power is above N_c, the solution will initially self-focus as in the unperturbed critical NLS. As a result, near the time of blow up in the absence of the perturbation, Conditions 2.1–2.2 will also be satisfied.

It is worth pointing out that, as studied in [12], various conservative perturbations of the critical NLS equation, for instance, self-focusing in fiber arrays (see [1–4, 16, 21] and references therein) and small dispersive fifth-power nonlinear perturbation to the classic NLS [18] have a generic form

$$f_1 \sim -\frac{C}{L^2}, \quad C = \text{constant}, \tag{2.15}$$

which results in a canonical focusing–defocusing oscillation.

Next, we shall derive the reduced equations (2.14) that correspond with the nonlinear Schrödinger–Helmholtz regularization system in the critical case $\sigma = 1$, $N = 2$ (the reason for this restriction is that it allows us to compare with numerical simulations). In this case, Eq. (1.3) reads

$$iv_t + \triangle v + |v|^2 v + \alpha^2 v \triangle u = 0. \tag{2.16}$$

Comparing Eq. (2.16) with (2.5) we have $\epsilon = \alpha^2$ and

$$F(v) = v\triangle u, \quad u - \alpha^2 \triangle u = |v|^2. \tag{2.17}$$

We shall assume that the system (2.16) satisfies all three conditions, (2.6), (2.7), and (2.9). Because

$$\psi_R(x) = \frac{1}{L} R\left(\frac{x}{L}\right) \exp(iS), \tag{2.18}$$

we have

$$F(\psi_R)(x) = \psi_R(x) \triangle u_R, \tag{2.19}$$

where u_R satisfies

$$u_R(x) - \alpha^2 \triangle u_R(x) = |\psi_R(x)|^2 = \left|\frac{1}{L} R\left(\frac{x}{L}\right) \exp(iS)\right|^2 = \frac{1}{L^2}\left|R\left(\frac{x}{L}\right)\right|^2. \tag{2.20}$$

For a given function g, solution to the equation

$$(I - \alpha^2 \triangle) u(x) = g\left(\frac{x}{L}\right), \quad x \in \mathbb{R}^2 \tag{2.21}$$

is given by

$$u(x) = \left(B_{\frac{\alpha}{L}} * g\right)\left(\frac{x}{L}\right),$$

where $B_{\frac{\alpha}{L}}$ is the modified Bessel potential or the Green function corresponding with the Helmholtz operator (see, e.g., [7] for reference on Bessel potential)

$$B_{\frac{\alpha}{L}}(x) = \frac{1}{2\alpha^2}\int_0^\infty \frac{e^{-s}e^{-\frac{|x|^2}{4s(\alpha/L)^2}}}{s^{N/2}}ds. \tag{2.22}$$

If we let $g(\cdot) = \frac{1}{L^2}R(\cdot)^2$, then we can write the solution to Eq. (2.20) as

$$u_R(x) = \left(B_{\frac{\alpha}{L}} * \frac{1}{L^2}R^2\right)\left(\frac{x}{L}\right) = \frac{1}{L^2}\left(B_{\frac{\alpha}{L}} * R^2\right)\left(\frac{x}{L}\right) \tag{2.23}$$

$$= \frac{1}{L^2}\frac{1}{2\alpha^2}\int_{\mathbb{R}^2}\left(\int_0^\infty \frac{e^{-s}e^{-\frac{|x/L-y|^2}{4s(\alpha/L)^2}}}{s}ds\right)R^2(y)\,dy_1\,dy_2, \tag{2.24}$$

where $y = (y_1, y_2)$. Substituting the above into (2.19) and using (2.20), we get

$$F(\psi_R)(x) = \psi_R(x)\triangle u_R(x) = \frac{1}{\alpha^2}(u_R(x) - |\psi_R(x)|^2)\psi_R(x) \tag{2.25}$$

$$= \frac{1}{\alpha^2 L}R\left(\frac{x}{L}\right)\left[u_R(x) - \frac{1}{L^2}R^2\left(\frac{x}{L}\right)\right]e^{iS}. \tag{2.26}$$

Now we can calculate the term f_1

$$\begin{aligned}
f_1 &= \frac{L}{\pi}Re\int_{\mathbb{R}^2} F(\psi_R(x))\exp(-iS)(R(\rho) + \rho R_\rho)\,dx_1\,dx_2 \\
&= \frac{L}{\pi}\int_{\mathbb{R}^2}\frac{1}{\alpha^2 L}R(\rho)\left[u_R(x) - \frac{1}{L^2}R^2(\rho)\right][R(\rho) + \rho R_\rho]\,dx_1\,dx_2 \\
&= \frac{1}{\pi\alpha^2}\int_{\mathbb{R}^2} u_R(x)R(\rho)[R(\rho) + \rho R_\rho]\,dx_1\,dx_2 \\
&\quad - \frac{1}{\pi\alpha^2}\frac{1}{L^2}\int_{\mathbb{R}^2} R^3(\rho)[R(\rho) + \rho R_\rho]\,dx_1\,dx_2 \\
&= J_1 - J_2.
\end{aligned}$$

It is easy to see that the second integral J_2 is a constant that does not depend on L. Indeed, applying change of variables: $\xi_1 = \frac{x_1}{L}$, $\xi_2 = \frac{x_2}{L}$, as $\rho = \frac{r}{L} = \frac{\sqrt{x_1^2+x_2^2}}{L} = \sqrt{\xi_1^2 + \xi_2^2}$, we get

$$J_2 = \frac{1}{\pi\alpha^2}\int_{\mathbb{R}^2} R^3(\rho)(R(\rho) + \rho R_\rho(\rho))\,d\xi_1\,d\xi_2 = c_0, \tag{2.27}$$

where c_0 is a constant that does not depend on L.

Next, let us look at the first integral J_1. Plugging the result of (2.24) into J_1, we get

$$J_1 = \frac{1}{\pi\alpha^2}\int_{\mathbb{R}^2} u_R(x)R(\rho)(R(\rho)+\rho R_\rho)\,dx_1\,dx_2 \tag{2.28}$$

$$= \frac{1}{\pi\alpha^2}\int_{\mathbb{R}^2}\frac{1}{L^2}\frac{1}{2\alpha^2}\int_{\mathbb{R}^2}\left(\int_0^\infty \frac{e^{-s}e^{-\frac{|x/L-y|^2}{4s(\alpha/L)^2}}}{s}\,ds\right)$$
$$\times R^2(y)\,dy_1\,dy_2 R(\rho)(R(\rho)+\rho R_\rho)\,dx_1\,dx_2 \tag{2.29}$$

$$= \frac{1}{2\pi}\frac{1}{\alpha^4}\frac{1}{L^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_0^\infty \frac{e^{-s}e^{-\frac{|x/L-y|^2}{4s(\alpha/L)^2}}}{s}\,ds$$
$$\times R^2(y)\,dy_1\,dy_2\,R(\rho)(R(\rho)+\rho R_\rho)\,dx_1\,dx_2. \tag{2.30}$$

Change of variables: $\xi = (\xi_1,\xi_2) = \left(\frac{x_1}{L},\frac{x_2}{L}\right)$, then we will have

$$J_1 = \frac{1}{2\pi}\frac{1}{\alpha^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_0^\infty \frac{e^{-s}e^{-\frac{|\xi-y|^2}{4s(\alpha/L)^2}}}{s}\,ds\;R^2(y)\,dy_1\,dy_2\,R(\rho)(R(\rho)+\rho R_\rho)\,d\xi_1\,d\xi_2. \tag{2.31}$$

So f_1 can be written as

$$f_1 = \frac{1}{2\pi}\frac{1}{\alpha^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_0^\infty \frac{e^{-s}e^{-\frac{|\xi-y|^2}{4s(\alpha/L)^2}}}{s}\,ds\,R^2(y)\,dy_1\,dy_2\,R(\rho)(R(\rho)+\rho R_\rho)\,d\xi_1\,d\xi_2 - c_0, \tag{2.32}$$

where c_0 is a constant given in (2.27).

Plugging into (2.14), we have the reduced system for the Schrödinger–Helmholtz system (1.1)

$$-L^3L_{tt} = \beta_0 + \frac{\alpha^2}{M}f_1, \quad \beta_0 = \beta(0) - \frac{\alpha^2}{2M}f_1(0) \tag{2.33}$$

where f_1 is given in (2.32).

Now one needs to study the ordinary differential equation (ODE) (2.33) with f_1 given in (2.32). The explicit form of the function f_1 is much more complicated when compared with the generic form of (2.15) due to the nonlocal nature of our perturbation term (2.17). The idea is to show that for L small, this additional term on the right-hand side of the first equation of (2.33) will prevent the singularity formation, i.e., prevents L from tending to zero as time evolves. In the next section, we will investigate the ODE system (2.33) by approximating the function f_1.

3. A SIMPLIFIED REDUCED SYSTEM

Now, without solving $u_R(x)$ explicitly as we did above, we will try to approximate $u_R(x)$ by asymptotic expansion in terms of $\frac{\alpha}{L}$, for small values of $\frac{\alpha}{L}$, and further approximate the function f_1.

3.1. First-Order Approximation

Recall that from (1.1) we have (when $\sigma = 1$, $N = 2$)

$$u - \alpha^2 \triangle u = |v|^2$$

or we can write $u(x) = (I - \alpha^2\triangle)^{-1}|v(x)|^2$. When α is very small, we can formally write $u(x)$ in the first leading order term: $u(x) = |v(x)|^2 + \mathcal{O}(\alpha^2)$. Now for $\psi_R(x) = \frac{1}{L}R(\frac{x}{L})\exp(iS)$, we can similarly write $u_R(x)$ as

$$\begin{aligned} u_R(x) &= |\psi_R(x)|^2 + \mathcal{O}\left(\left(\frac{\alpha}{L}\right)^2\right) \\ &= \frac{1}{L^2}R^2\left(\frac{x}{L}\right) + \mathcal{O}\left(\left(\frac{\alpha}{L}\right)^2\right) \end{aligned} \tag{3.1}$$

so we have

$$\begin{aligned} F(\psi_R)(x) &= (\triangle_x u_R(x))\psi_R(x) \\ &\sim (\triangle_x(|\psi_R(x)|^2))\psi_R(x) \\ &\sim \frac{1}{L^2}\left(\triangle_x R^2\left(\frac{x}{L}\right)\right)\frac{1}{L}R\left(\frac{x}{L}\right)\exp(iS). \end{aligned} \tag{3.2}$$

Substituting this into the equation for f_1 (2.11), we get

$$\begin{aligned} f_1 &= \frac{L}{\pi} Re \int_{\mathbb{R}^2} F(\psi_R(x))\exp(-iS)(R(\rho) + \rho R_\rho)\,dx_1\,dx_2 \\ &\sim \frac{L}{\pi}\int_{\mathbb{R}^2}\frac{1}{L^2}\left(\triangle_x R^2\left(\frac{x}{L}\right)\right)\frac{1}{L}R\left(\frac{x}{L}\right)(R(\rho) + \rho R_\rho)\,dx_1\,dx_2. \end{aligned} \tag{3.3}$$

Next, we make change of variables:

$$\xi = \frac{x}{L}, \quad \xi = (\xi_1, \xi_2)$$

then for $\rho = |\xi|$, and by the chain rule, we get

$$\begin{aligned} f_1 &\sim \frac{1}{\pi}\frac{1}{L^2}\int_{\mathbb{R}^2}(\triangle_\xi R^2(\rho))R(\rho)(R(\rho) + \rho R_\rho)\,d\xi_1\,d\xi_2 \\ &\sim -\frac{C_1}{L^2} = I_1 \end{aligned} \tag{3.4}$$

where

$$C_1 = -\frac{1}{\pi}\int_{\mathbb{R}^2}(\triangle_\xi R^2)R(R+\rho R_\rho)\,d\xi_1\,d\xi_2$$
$$= 2\int_0^\infty [(R^2)_\rho]^2\rho\,d\rho > 0, \tag{3.5}$$

where the detail of the calculation of the above integral is presented in Claim A.1 of the Appendix.

Plugging into (2.33), we have the leading order of the reduced system, which turns out to be of the generic form

$$-L^3L_{tt} = \beta_0 + \frac{\alpha^2}{2M}f_1 \tag{3.6}$$

with $f_1 \sim -\frac{C_1}{L^2}$ and $\beta_0 = \beta(0) + \frac{\alpha^2 C_1}{2M}\frac{1}{L^2(0)} \ll 1$ as $\beta(0) \ll 1$, $\frac{\alpha}{L} \ll 1$. Fibich and Papanicolaou [12] showed that there is no singularity in finite time with this perturbation. In fact, substituting f_1 (3.4) into the above equation, we get

$$-L^3L_{tt} = \beta_0 - \frac{C_1}{2M}\frac{\alpha^2}{L^2}. \tag{3.7}$$

Write $y = L^2$, then y satisfies the following equation:

$$(y_t)^2 = 4\beta_0 - \frac{\alpha^2 C_1}{M}\frac{1}{y} + 4D_0 y, \tag{3.8}$$

where D_0 is a constant satisfying $D_0 = L_t^2(0) - \frac{\beta_0}{L^2(0)} + \frac{\alpha^2 C_1}{4M}\frac{1}{L^4(0)}$.

Because $\frac{\alpha^2 C_1}{M} > 0$, y cannot go to zero in the above equation, i.e., L cannot go to zero, which explains the prevention of the singularity formation, at this leading order in the expansion.

3.2. Second-Order Approximation

In the previous subsection, we use the asymptotic expansion by taking the first leading term as approximation for the solution u of the Helmholtz equation $u - \alpha^2\triangle u = |v|^2$. One might naturally ask whether we will get better approximation and still have the no blow up structure if we approximate the solution of the Helmholtz equation $u(x)$ by taking one more leading term. To investigate this, we proceed similarly as before: we can formally write $u(x)$ in the first two leading terms: $u(x) = |v(x)|^2 + \alpha^2\triangle|v(x)|^2 + \mathcal{O}(\alpha^4)$, so we have

$$u_R(x) = |\psi_R(x)|^2 + \alpha^2\triangle_x|\psi_R(x)|^2 + \mathcal{O}\left(\left(\frac{\alpha}{L}\right)^4\right)$$

$$
\begin{aligned}
&= \frac{1}{L^2}R^2\left(\frac{x}{L}\right) + \alpha^2\triangle_x\left(\frac{1}{L^2}R^2\left(\frac{x}{L}\right)\right) + \mathcal{O}\left(\left(\frac{\alpha}{L}\right)^4\right) \\
&\sim \frac{1}{L^2}R^2\left(\frac{x}{L}\right) + \left(\frac{\alpha}{L}\right)^2\triangle_x R^2\left(\frac{x}{L}\right) \qquad (3.9)
\end{aligned}
$$

and then

$$
\begin{aligned}
F(\psi_R)(x) &= (\triangle_x u_R(x))\psi_R(x) \\
&\sim \left(\triangle_x\left(\frac{1}{L^2}R^2\left(\frac{x}{L}\right) + \left(\frac{\alpha}{L}\right)^2\triangle_x R^2\left(\frac{x}{L}\right)\right)\right)\frac{1}{L}R\left(\frac{x}{L}\right)\exp(iS) \\
&\sim \left(\frac{1}{L^2}\triangle_x R^2\left(\frac{x}{L}\right) + \left(\frac{\alpha}{L}\right)^2\triangle_x^2 R^2\left(\frac{x}{L}\right)\right)\frac{1}{L}R\left(\frac{x}{L}\right)\exp(iS).
\end{aligned}
$$

Substituting this into the equation of f_1 (2.11), we get

$$
\begin{aligned}
f_1 &= \frac{L}{\pi}Re\int_{\mathbb{R}^2} F(\psi_R(x))\exp(-iS)(R(\rho)+\rho R_\rho)\,dx_1\,dx_2 \\
&\sim \frac{L}{\pi}\int_{\mathbb{R}^2}\left(\frac{1}{L^2}\triangle_x R^2\left(\frac{x}{L}\right) + \left(\frac{\alpha}{L}\right)^2\triangle_x^2 R^2\left(\frac{x}{L}\right)\right)\frac{1}{L}R\left(\frac{x}{L}\right)(R(\rho)+\rho R_\rho)\,dx_1\,dx_2 \\
&\sim \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{1}{L^2}\left(\triangle_x R^2\left(\frac{x}{L}\right)\right)R\left(\frac{x}{L}\right)(R(\rho)+\rho R_\rho)\,dx_1\,dx_2 \\
&\quad + \frac{1}{\pi}\int_{\mathbb{R}^2}\left(\frac{\alpha}{L}\right)^2\triangle_x^2 R^2\left(\frac{x}{L}\right)R\left(\frac{x}{L}\right)(R(\rho)+\rho R_\rho)\,dx_1\,dx_2 \\
&\sim I_1 + I_2.
\end{aligned}
$$

Here the first integral I_1 is equal to $-\frac{C_1}{L^2}$ with C_1 in (3.5) as we calculated in the first-order expansion case. For the second integral I_2, after change of variables $\xi = \frac{x}{L}$, $\xi = (\xi_1, \xi_2)$ and $\rho = |\xi|$, it gives

$$
\begin{aligned}
I_2 &= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\triangle_\xi^2(R^2))R(R+\rho R_\rho)\,d\xi_1\,d\xi_2 \\
&= \frac{\alpha^2 C_2}{L^4}, \qquad (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
C_2 &= \frac{1}{\pi}\int_{\mathbb{R}^2}(\triangle_\xi^2(R^2))R(R+\rho R_\rho)\,d\xi_1\,d\xi_2 \\
&= \frac{3}{2\pi}\int_{\mathbb{R}^2}(\triangle_\xi R^2)^2\,d\xi_1\,d\xi_2 > 0, \qquad (3.11)
\end{aligned}
$$

where the detailed calculation of the above integral is presented in Claim A.2 of the Appendix.

Therefore, we obtain

$$f_1 \sim -\frac{C_1}{L^2} + \frac{C_2\alpha^2}{L^4} \tag{3.12}$$

with a correction next-order term to the generic form (2.15). It yields the reduced equation of the leading order

$$-L^3 L_{tt} = \beta_0 + \frac{\alpha^2}{2M} f_1. \tag{3.13}$$

Substituting f_1 into the above Equation (3.13), we get

$$-L^3 L_{tt} = \beta_0 - \frac{C_1}{2M}\left(\frac{\alpha}{L}\right)^2 + \frac{C_2}{2M}\left(\frac{\alpha}{L}\right)^4. \tag{3.14}$$

Then $y = L^2$ satisfies the following equation:

$$(y_t)^2 = 4\beta_0 - \frac{\alpha^2 C_1}{M}\frac{1}{y} + \frac{2}{3}\frac{\alpha^4 C_2}{M}\frac{1}{y^2} + E_0 y, \tag{3.15}$$

where E_0 is a constant satisfying $E_0 = 4L_t^2(0) - \frac{4\beta_0}{L(0)^2} + \frac{C_1}{M}\frac{\alpha^2}{L^4(0)} - \frac{2C_2}{3M}\frac{\alpha^4}{L^6(0)}$.

Let us rewrite the right-hand side of the above Equation (3.15) by substituting $y = L^2$ and E_0 into the equation

$$(y_t)^2 = 4L_t^2(0) + 4\beta_0\left(1 - \left(\frac{L(t)}{L(0)}\right)^2\right) - \frac{C_1}{M}\left(\frac{\alpha}{L(t)}\right)^2\left(1 - \left(\frac{L(t)}{L(0)}\right)^4\right) + \frac{2C_2}{3M}\left(\frac{\alpha}{L(t)}\right)^4\left(1 - \left(\frac{L(t)}{L(0)}\right)^6\right).$$

When L approaches zero, $\frac{L(t)}{L(0)}$ is very small, and in the regime of $\frac{\alpha}{L(t)} \ll 1$, which is a required assumption for the expansion, the right-hand side will remain positive when $L_t(0)$ is large. In other words, contrary to the equation of first-order expansion (3.8), the second-order expansion Equation (3.14) might blow up with certain large $L_t(0)$; we will see this is also verified in the numerical computation in the next section.

3.3. Third-Order Expansion

With finite time blow up in the second-order expansion, one might try to include a higher order term in the expansion of the solution of the

Helmholtz equation:

$$u(x) = |v(x)|^2 + \alpha^2 \triangle |v(x)|^2 + \alpha^4 \triangle^2 |v(x)|^2 + \mathcal{O}(\alpha^6). \tag{3.16}$$

As a result, we have

$$u_R(x) = \frac{1}{L^2} R^2\left(\frac{x}{L}\right) + \left(\frac{\alpha}{L}\right)^2 \triangle_x R^2\left(\frac{x}{L}\right) + \left(\frac{\alpha}{L}\right)^4 \triangle_x^2 R^2\left(\frac{x}{L}\right) + \mathcal{O}\left(\left(\frac{\alpha}{L}\right)^6\right)$$

and then

$$\begin{aligned} F(\psi_R)(x) &= (\triangle_x u_R(x))\psi_R(x) \\ &\sim \left(\frac{1}{L^2}\triangle_x R^2\left(\frac{x}{L}\right) + \frac{\alpha^2}{L^2}\triangle_x^2 R^2\left(\frac{x}{L}\right) + \frac{\alpha^4}{L^4}\triangle_x^3 R^2\left(\frac{x}{L}\right)\right)\frac{1}{L}R\left(\frac{x}{L}\right)\exp(iS). \end{aligned}$$

Now f_1 satisfies the following expression

$$\begin{aligned} f_1 &= \frac{L}{\pi} Re \int_{\mathbb{R}^2} F(\psi_R(x)) \exp(-iS)(R(\rho) + \rho R_\rho)\, dx_1\, dx_2 \\ &\sim I_1 + I_2 + I_3, \end{aligned}$$

where I_1, I_2 are the same as in the first-order and second-order expansions (3.4) and (3.10), and we have

$$\begin{aligned} I_3 &= \frac{L}{\pi}\int_{\mathbb{R}^2} \frac{\alpha^4}{L^2}\left(\triangle_x^3 R^2\left(\frac{x}{L}\right)\right)\frac{1}{L}R\left(\frac{x}{L}\right)(R(\rho)+\rho R_\rho)\, dx_1\, dx_2 \\ &= \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2} (\triangle_\xi^3 R^2) R (R + \rho R_\rho)\, d\xi_1\, d\xi_2 \\ &= -\frac{\alpha^4 C_3}{L^6}, \end{aligned} \tag{3.17}$$

where the constant

$$C_3 = \frac{2}{\pi}\int_{\mathbb{R}^2} (\nabla \triangle R^2)^2 d\xi_1 d\xi_2 > 0, \quad \xi = \frac{x}{L} = (\xi_1, \xi_2), \tag{3.18}$$

where the detailed calculation is presented in Claim A.3 of the Appendix.

Substituting f_1 into the reduced equation of L (3.6), we get

$$-L^3 L_{tt} = \beta_0 - \frac{C_1}{2M}\frac{\alpha^2}{L^2} + \frac{C_2}{2M}\frac{\alpha^4}{L^4} - \frac{C_3}{2M}\frac{\alpha^6}{L^6}. \tag{3.19}$$

So $y = L^2$ satisfies the following equation

$$(y_t)^2 = 4\beta_0 - \frac{\alpha^2 C_1}{M}\frac{1}{y} + \frac{2}{3}\frac{\alpha^4 C_2}{M}\frac{1}{y^2} - \frac{\alpha^6 C_3}{2M}\frac{1}{y^3} + F_0 y, \tag{3.20}$$

where $F_0 = 4L_t^2(0) - \frac{4\beta_0}{L^2(0)} + \frac{\alpha^2 C_1}{M}\frac{1}{L^4(0)} - \frac{2}{3}\frac{\alpha^4 C_2}{M}\frac{1}{L^6(0)} + \frac{\alpha^6 C_3}{2M}\frac{1}{L^8(0)}$. In this equation, because C_3 is positive, we can see again that y cannot approach zero, i.e., L cannot go to zero, which prevents singularity formation.

4. NUMERICAL RESULTS

In this section, we will show some numerical results of the evolution of $L(t)$ in three expansion cases, (3.7), (3.14) and (3.19), in order to study the prevention of blow up of the Schrödinger–Helmholtz system. We will first consider the first-order expansion case.

4.1. First-Order Expansion

Let us look at the Equation (3.7). After some algebraic calculation and calculus integration, we come up with Equation (3.8):

$$(y_t)^2 = 4\beta_0 - \frac{\alpha^2 C_1}{M}\frac{1}{y} + 4D_0 y,$$

where $\beta_0 = \beta(0) + \frac{C_1}{2M}\frac{\alpha^2}{L^2(0)}$ and $D_0 = L_t^2(0) - \frac{\beta_0}{L^2(0)} + \frac{C_1}{4M}\frac{\alpha^2}{L^4(0)}$.

From above we know that y cannot approach zero, equivalently, L cannot go to zero, which prevents singularity formation.

Furthermore, Fibich and Papanicolaou [12] derived the generic equation

$$(y_t)^2 = 4\beta_0 - \frac{\alpha^2 C_1}{M}\frac{1}{y} + 4\frac{H_0}{M}y = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - ym),$$

where

$$y_M = \frac{\sqrt{\beta_0^2 + \alpha^2 C_1 H_0/M^2} + \beta_0}{-2H_0/M} = \frac{M\beta_0}{-H_0}\left[1 + \mathcal{O}\left(\frac{\alpha^2 H_0}{\beta_0^2}\right)\right],$$
$$y_m = \frac{\alpha^2 C_1}{2M}\frac{1}{\sqrt{\beta_0^2 + \alpha^2 C_1 H_0/M^2} + \beta_0} = \frac{\alpha^2 C_1}{4M\beta_0}\left[1 + \mathcal{O}\left(\frac{\alpha^2 H_0}{\beta_0^2}\right)\right],$$
$$H_0 = H(0) + \frac{\alpha^2 C_1}{4}\frac{1}{L^4(0)}.$$

From above we see that when $H_0 > 0$, and $L_t(0) > 0$, L is monotonically defocusing to infinity; when $H_0 > 0$ and $L_t(0) < 0$, self-focusing is arrested when $L = L_m = (y_m)^{1/2} > 0$, after which L is monotonically defocusing to infinity; when $H_0 < 0$, then L goes through periodic oscillation between $L_m = (y_m)^{1/2}$ and $L_M = (y_M)^{1/2}$ (see Fig. 1).

4.2. Second-Order Expansion

Now we look at the Equation (3.14) of next-order expansion:

$$-L^3 L_{tt} = \left(\frac{C_2}{2M}\right)\left(\frac{\alpha}{L}\right)^4 - \left(\frac{C_1}{2M}\right)\left(\frac{\alpha}{L}\right)^2 + \beta_0. \tag{4.1}$$

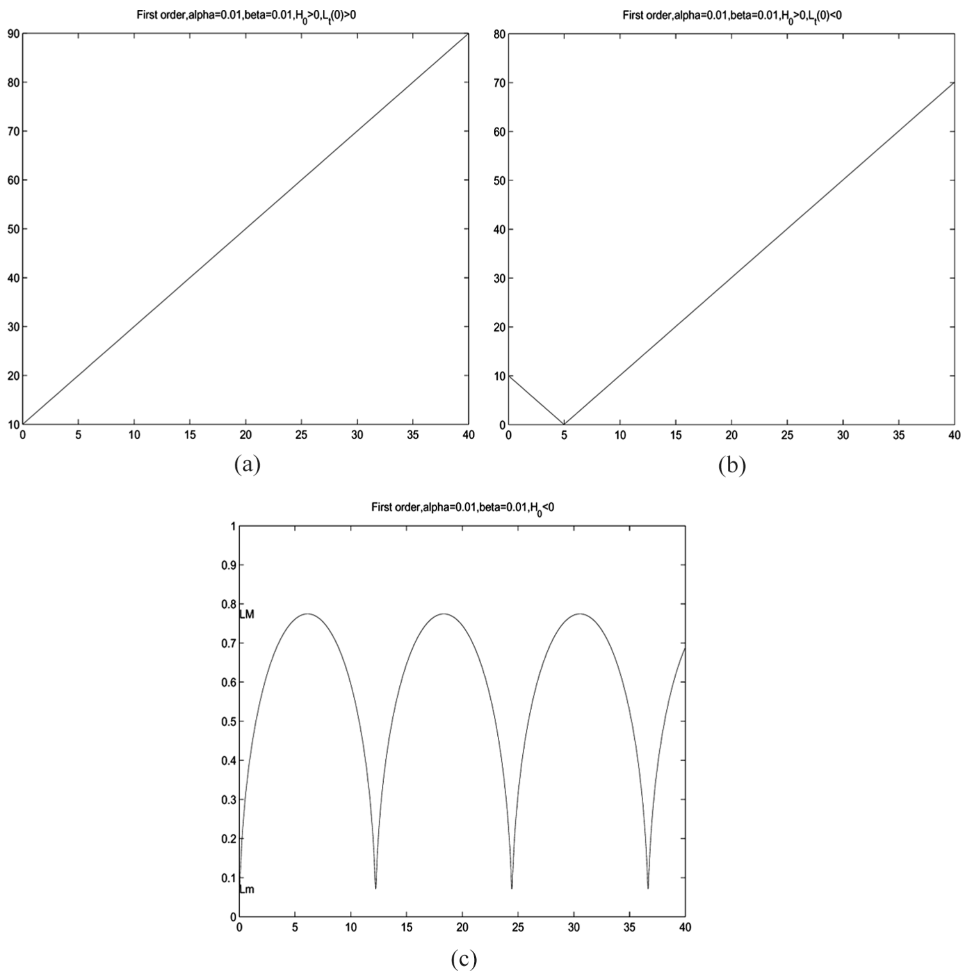


FIGURE 1 L evolves in time in first-order expansion: (a) Monotonic defocusing, $H_0 > 0, L_t(0) > 0$; (b) First focusing then defocusing, $H_0 > 0, L_t(0) < 0$; (c) Oscillation, $H_0 < 0$. For all cases, $\alpha = 0.01$, $\beta_0 = 0.01$. In (a), (b), $\frac{\alpha}{L(0)} = 0.001$. In (c), $\frac{\alpha}{L(0)} = 1/8$.

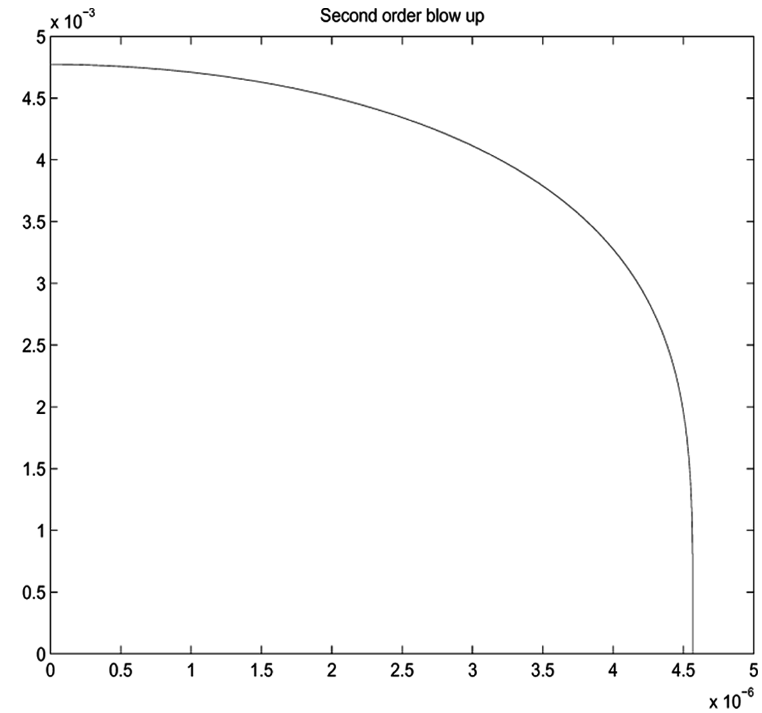


FIGURE 2 Flow of L in second-order expansion. L decreases to zero in finite time, $\alpha = 0.01, \beta_0 \gg 1$, $L(0) = 0.1, L_t(0) = -2$.

In this equation, when $\beta_0 > \frac{C_1^2}{C_2}\frac{1}{8M}$, the right-hand side is definitely positive, then for any initial data $L(0)$ with $L_t(0) < 0$, the solution L will monotonically decrease and approach zero in finite time. In other words, when the initial excess energy is larger than certain amount $\left(\frac{C_1^2}{C_2}\frac{1}{8M}\right)$ and L is initially focusing, L will focus and blow up in finite time (see Fig. 2). However, this is not valid to begin with applying the modulation theory. Recall that for us to apply the modulation theory to a perturbed critical NLS, we require three conditions to hold. One of the conditions is to require $|\beta(t)| \ll 1$ (see 2.7 or 2.8).

So we will consider the case when $0 < \beta_0 < \frac{C_1^2}{C_2}\frac{1}{8M}$, then we have $r_{\text{low}} = \sqrt{\frac{C_1 - 2M\sqrt{K}}{2C_2}}$, $r_{\text{high}} = \sqrt{\frac{C_1 + 2M\sqrt{K}}{2C_2}}$, where $K = \frac{C_1^2}{C_2}\frac{1}{8M} - \beta_0$. In this case, when initially $\frac{\alpha}{L(0)} > r_{\text{high}}$ and $L_t(0) < 0$, the right-hand side of the Equation (4.1) will remain positive, so L will monotonically decrease to zero, which is similar to the case of $\beta_0 > \frac{C_1^2}{C_2}\frac{1}{8M}$. Once again, this is not valid here for the discussion because the asymptotic expansion (3.9) is valid under the assumption that $\frac{\alpha}{L} \ll 1$, so we need only to consider the situation of $\frac{\alpha}{L(0)}$ small, in this case, $\frac{\alpha}{L(0)} < r_{\text{high}}$.

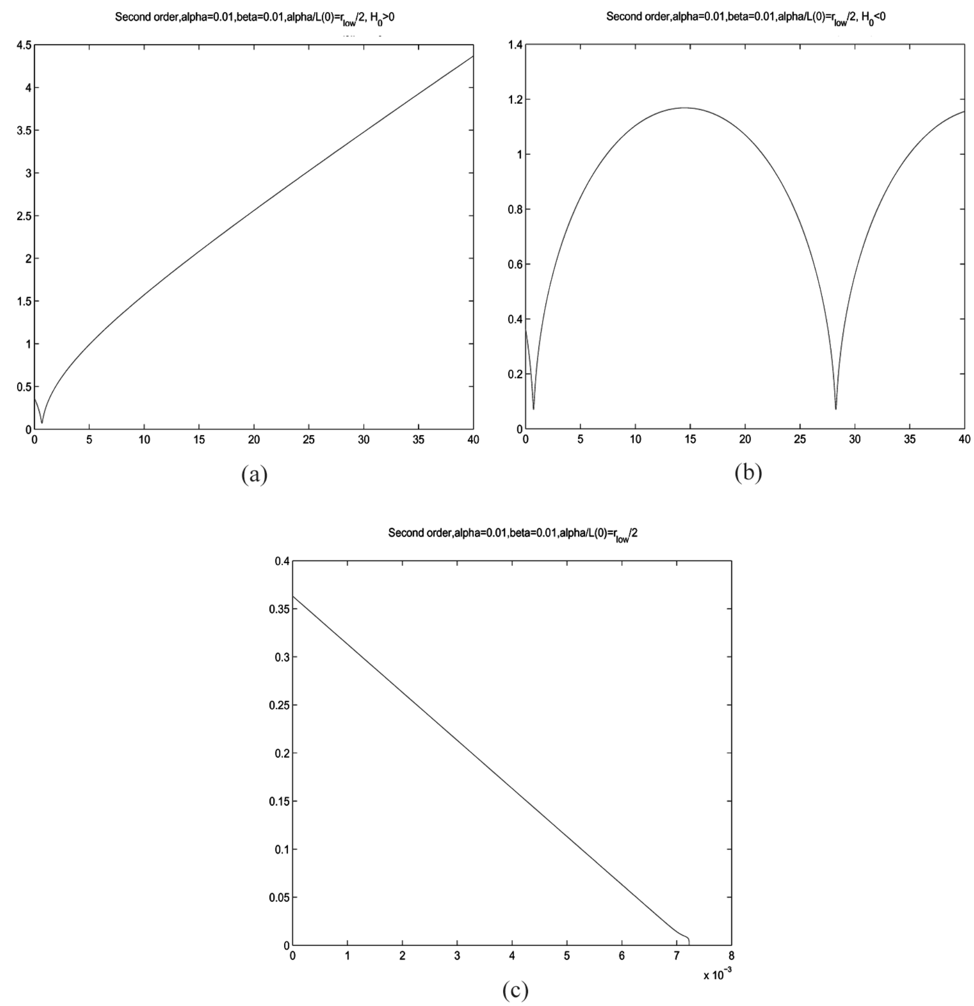
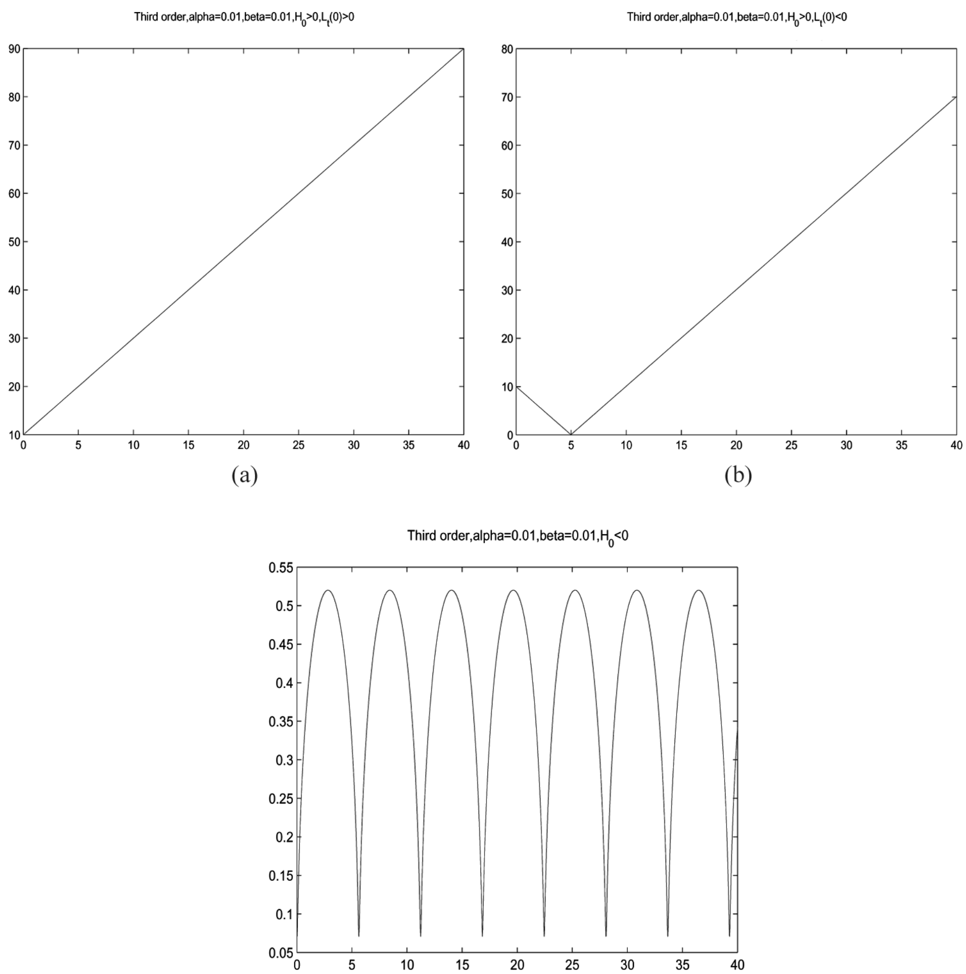


FIGURE 3 Flow of L in second-order expansion: (a) L defocuses to infinity when $H_0 > 0$. (b) L oscillates between two values when $H_0 < 0$. (c) L blows up when $|L_t(0)|$ is large. For all cases, we use $\alpha = 0.01, \beta_0 = 0.01 \ll \frac{C_1^2}{C_2}\frac{1}{8M}, \alpha/L(0) = r_{\text{low}}/2$.

Finally, we consider only the case $0 < \beta_0 < \frac{C_1^2}{C_2}\frac{1}{8M}$ and $0 < \frac{\alpha}{L(0)} < r_{\text{high}}$.

First we focus on $0 < \frac{\alpha}{L(0)} < r_{\text{low}}$. In this case, L might defocus to infinity, oscillate between two values, or even blow up in finite time depending on different initial condition of $L_t(0)$ for given β_0, α and $L(0)$. In Fig. 3, we take the parameters $\alpha = 0.01$, $\frac{\alpha}{L(0)} = r_{\text{low}}/2$ and $\beta_0 = 0.01$. When initially $L_t(0) > -44.9999$, L will eventually defocus to infinity if $H_0 > 0$ (Fig. 3(a)) and L will oscillate between two values if $H_0 < 0$ (Fig. 3(b)); when initially $L_t(0) < -49.9999$, L will approach zero in finite time, i.e., we observe singularity in finite time (Fig. 3(c)). This numerical result is expected from analysis at the end of Subsection 3.2.

Similarly, for $\eta_{\text{low}} < \frac{\alpha}{L(0)} < \eta_{\text{high}}$, we will observe different behaviors—defocusing, oscillation, or focusing depending on $L_t(0)$ and H_0. For instance, when $\frac{\alpha}{L(0)} = \frac{\eta_{\text{low}}+\eta_{\text{high}}}{2}$, we have the threshold value $L_t^c = -39.9999$, i.e., when $L_t(0) > -39.9999$, L will eventually defocus to infinity if $H_0 > 0$ and oscillate between two values if $H_0 < 0$; when $L_t(0) < -39.9999$, L will eventually decrease to zero.



(a) (b) (c)

FIGURE 4 Flow of L in third-order expansion: (a) Monotonic defocusing, $H_0 > 0, L_t(0) > 0$. (b) First focusing then defocusing, $H_0 > 0, L_t(0) < 0$. (c) Oscillation between two values, $H_0 < 0$. For all cases, $\alpha = 0.01, \beta_0 = 0.01$. In (a), (b), $\frac{\alpha}{L(0)} = 0.001$. In (c), $\frac{\alpha}{L(0)} = 1/8$.

4.3. Third-Order Expansion

Lastly we study the Equation (3.19) of higher order expansion:

$$-L^3 L_{tt} = \beta_0 - \frac{C_1}{2M}\frac{\alpha^2}{L^2} + \frac{C_2}{2M}\frac{\alpha^4}{L^4} - \frac{C_3}{2M}\frac{\alpha^6}{L^6}.$$

By defining $y = L^2$ and integrating the equation, we get equation (3.20)

$$(y_t)^2 = 4\beta_0 - \frac{\alpha^2 C_1}{M}\frac{1}{y} + \frac{2}{3}\frac{\alpha^4 C_2}{M}\frac{1}{y^2} - \frac{\alpha^6 C_3}{2M}\frac{1}{y^3} + F_0 y,$$

where $F_0 = 4L_t^2(0) - \frac{4\beta_0}{L^2(0)} + \frac{\alpha^2 C_1}{M}\frac{1}{L^4(0)} - \frac{2}{3}\frac{\alpha^4 C_2}{M}\frac{1}{L^6(0)} + \frac{\alpha^6 C_3}{2M}\frac{1}{L^8(0)}$.

In the above Equation (3.20), y cannot approach zero as the leading order term on the right-hand side is of negative sign when y goes to zero, equivalently, L cannot approach zero. The nature of this equation is the same as that of Equation (3.8), and we see the same pattern in the numerical result (see Fig. 4).

5. CONCLUSIONS

From the analysis and numerical computation, we see that the regularization of the classic NLS effectively prevents singularity formation with positive parameter $\alpha > 0$. By asymptotically expanding the solution of the Helmholtz equation to approximate the reduced system of the modulation theory, we observe strong no blow up pattern in both first-order and third-order expansion. In the valid regime of the expansion and modulation theory, we also observe no blow up pattern in the second-order expansion with further restriction on certain condition: $L_t(0) > L_t^c$, threshold initial value of $L_t(0)$. This phenomenon is expected for even higher order expansion, say fourth-order expansion approximation. One of the reasons is that the Laplace operator is not bounded, which causes instability for the expansion of the solution of the Helmholtz equation.

APPENDIX

For completeness, we present in this section the detail of the calculation of the integrals (3.4), (3.10), and (3.17).

Claim A.1. *The integral I_1 in (3.4) for the first-order expansion can be simplified as*

$$I_1 = \frac{1}{\pi}\frac{1}{L^2}\int_{\mathbb{R}^2}(\triangle_\xi R^2)R(R+\rho R_\rho)\,d\xi_1\,d\xi_2 = -\frac{2}{L^2}\int_0^\infty [(R^2)_\rho]^2\rho\,d\rho.$$

Proof.

$$\begin{aligned}
I_1 &= \frac{1}{\pi}\frac{1}{L^2}\int_{\mathbb{R}^2}(\triangle_\xi R^2(\rho))R(\rho)(R(\rho)+\rho R_\rho)\,d\xi_1\,d\xi_2\\
&= \frac{1}{\pi}\frac{1}{L^2}\int_{\mathbb{R}^2}(\triangle R^2)R^2\,d\xi_1\,d\xi_2 + \frac{1}{\pi}\frac{1}{L^2}\int_{\mathbb{R}^2}(\triangle R^2)RR_\rho\rho\,d\xi_1\,d\xi_2\\
&= I_{11}+I_{12}.
\end{aligned}$$

In the rest of this section, all the integrands and integrals are of variables ξ or ρ (i.e., they are the scaled variables) unless it is stated otherwise.

Now for I_{11}, we integrate by parts once and change the variables by $\xi_1 = \rho\cos\theta$, $\xi_2 = \rho\sin\theta$; we obtain

$$\begin{aligned}
I_{11} &= -\frac{1}{\pi}\frac{1}{L^2}\int_{\mathbb{R}^2}(\nabla R^2)^2\,d\xi_1\,d\xi_2\\
&= -\frac{1}{\pi}\frac{1}{L^2}2\pi\int_0^\infty[(R^2)_\rho]^2\rho\,d\rho\\
&= -\frac{2}{L^2}\int_0^\infty[(R^2)_\rho]^2\rho\,d\rho,
\end{aligned}$$

which gives us exactly the constant C_1 as in (3.5).

For I_{12}, we show now it is identically zero.

Using polar coordinates, we get

$$\begin{aligned}
I_{12} &= \frac{1}{\pi}\frac{1}{L^2}2\pi\int_0^\infty\left((R^2)_{\rho\rho}+\frac{1}{\rho}(R^2)_\rho\right)RR_\rho\rho^2\,d\rho\\
&= \frac{2}{L^2}\int_0^\infty(R^2)_{\rho\rho}RR_\rho\,\rho^2\,d\rho + \frac{2}{L^2}\int_0^\infty(R^2)_\rho RR_\rho\,\rho\,d\rho\\
&= \frac{1}{L^2}\int_0^\infty(R^2)_{\rho\rho}(R^2)_\rho\,\rho^2\,d\rho + \frac{1}{L^2}\int_0^\infty(R^2)_\rho(R^2)_\rho\rho\,d\rho\\
&= X+Y.
\end{aligned}$$

For the first integral X, after rewriting and integration by parts once, we get

$$\begin{aligned}
X &= \frac{1}{L^2}\int_0^\infty\frac{1}{2}[((R^2)_\rho)^2]_\rho\rho^2\,d\rho\\
&= -\frac{1}{L^2}\int_0^\infty((R^2)_\rho)^2\rho\,d\rho\\
&= -Y.
\end{aligned}$$

We conclude that $I_{12}=0$, which yields the result of (3.4) and (3.5). $\square$

Claim A.2. *The integral I_2 (3.10) in the next-order expansion can be simplified as*

$$I_2 = \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\triangle_\xi^2(R^2))R(R+\rho R_\rho)d\xi_1\,d\xi_2 = \frac{3\alpha^2}{2\pi L^4}\int_{\mathbb{R}^2}(\triangle_\xi R^2)^2 d\xi_1\,d\xi_2.$$

Proof.

$$\begin{aligned}
I_2 &= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\triangle^2(R^2))R(R+\rho R_\rho)d\xi_1\,d\xi_2\\
&= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\triangle^2(R^2))R^2 d\xi_1\,d\xi_2 + \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\triangle^2(R^2))R(\rho R_\rho)d\xi_1\,d\xi_2\\
&= I_{21} + I_{22}.
\end{aligned}$$

After integration by parts twice, we get

$$I_{21} = \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}[\triangle(R^2)]^2 d\xi_1\,d\xi_2.$$

Next we will show that $I_{22} = \frac{1}{2}I_{21}$, which yields (3.11).

Recall that $\rho R_\rho = \xi\cdot\nabla R$, $\xi = (\xi_1, \xi_2)$, $\rho^2 = \xi_1^2 + \xi_2^2$, so we have

$$\begin{aligned}
I_{22} &= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\triangle^2(R^2))(R(\xi\cdot\nabla R))d\xi\\
&= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\triangle^2(R^2))\left(\xi\cdot\nabla\frac{1}{2}R^2\right)d\xi\\
&= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}\triangle(R^2)\triangle\left(\xi\cdot\nabla\frac{1}{2}R^2\right)d\xi\\
&= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}\triangle(R^2)\left(\triangle(R^2)+\xi\cdot\nabla\left(\triangle\frac{1}{2}R^2\right)\right)d\xi\\
&= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}[\triangle(R^2)]^2 d\xi + \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}\triangle(R^2)\left(\xi\cdot\nabla\left(\triangle\frac{1}{2}R^2\right)\right)d\xi\\
&= P + Q.
\end{aligned}$$

For the second integral in the last line, we rewrite it then integrate by parts and obtain

$$\begin{aligned}
Q &= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}\triangle(R^2)\left(\xi\cdot\nabla\left(\triangle\frac{1}{2}R^2\right)\right)d\xi\\
&= \frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}\xi\cdot\nabla\left(\frac{1}{4}(\triangle(R^2))^2\right)d\xi
\end{aligned}$$

$$= -\frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}(\nabla\cdot\xi)\frac{1}{4}(\triangle(R^2))^2 d\xi$$
$$= -\frac{1}{\pi}\frac{\alpha^2}{L^4}\int_{\mathbb{R}^2}\frac{1}{2}[\triangle(R^2)]^2 d\xi$$
$$= -\frac{1}{2}P.$$

So we get $I_{22} = P + Q = \frac{1}{2}P = \frac{1}{2}I_{21}$, so we conclude that $I_2 = \frac{3}{2}I_{21} = \frac{\alpha^2}{L^4}\frac{3}{2\pi}\int_{\mathbb{R}^2}[\triangle(R^2)]^2 d\xi$, which yields exactly (3.10) and (3.11). □

Claim A.3. *The integral I_3 (3.17) in the calculation of higher order expansion can be simplified as*

$$I_3 = \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle_\xi^3 R^2)R(R+\rho R_\rho)d\xi_1\, d\xi_2 = -\frac{2}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\nabla\triangle R^2)^2 d\xi_1\, d\xi_2.$$

Proof.

$$I_3 = \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^3(R^2))R(R+\rho R_\rho)d\xi$$
$$= \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^3(R^2))R^2 d\xi + \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^3(R^2))R(\rho R_\rho)d\xi$$
$$= I_{31} + I_{32}.$$

After integration by parts three times, we get

$$I_{31} = -\frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\nabla\triangle(R^2))^2 d\xi.$$

Next, we will show that $I_{32} = I_{31} = -\frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\nabla\triangle(R^2))^2 d\xi$.

$$I_{32} = \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^3(R^2))R(\rho R_\rho)d\xi$$
$$= \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^3(R^2))R(\xi\cdot\nabla R)d\xi$$
$$= \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^3(R^2))\left(\xi\cdot\nabla\frac{1}{2}R^2\right)d\xi$$
$$= \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^2(R^2))\triangle\left(\xi\cdot\nabla\frac{1}{2}R^2\right)d\xi$$

$$
\begin{aligned}
&= \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^2(R^2))\left(\triangle(R^2)+\xi\cdot\nabla\left(\triangle\frac{1}{2}R^2\right)\right)d\xi\\
&= \frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\triangle^2(R^2))\triangle(R^2)\,d\xi+\frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}\frac{1}{2}(\triangle^2(R^2))(\xi\cdot\nabla(\triangle R^2))\,d\xi\\
&= A+B.
\end{aligned}
$$

For A, after integration by parts once, we obtain

$$
A=-\frac{1}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\nabla\triangle(R^2))^2\,d\xi.
$$

For B, we define $\phi=\triangle(R^2)$, a scalar function. Then we rewrite and calculate the term B as follows

$$
\begin{aligned}
B &= \frac{1}{2\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}\triangle\phi(\xi\cdot\nabla\phi)\,d\xi\\
&= \frac{1}{2\pi}\frac{\alpha^4}{L^6}2\pi\int_0^\infty\left(\phi_{\rho\rho}+\frac{1}{\rho}\phi_\rho\right)(\rho\phi_\rho)\rho\,d\rho\\
&= \frac{\alpha^4}{L^6}\int_0^\infty\phi_{\rho\rho}\phi_\rho\rho^2+(\phi_\rho)^2\rho\,d\rho\\
&= \frac{\alpha^4}{L^6}\int_0^\infty\frac{1}{2}((\phi_\rho)^2\rho^2)_\rho\,d\rho\\
&= 0.
\end{aligned}
$$

So we have now $I_3=I_{31}+I_{32}=2I_{31}=-\frac{2}{\pi}\frac{\alpha^4}{L^6}\int_{\mathbb{R}^2}(\nabla\triangle(R^2))^2\,d\xi$, which concludes our result of (3.17) and (3.18). □

ACKNOWLEDGMENTS

We would like to thank Professor Gadi Fibich for the valuable comments and suggestion. This work was supported in part by NSF grants no. DMS-0504619 and no. DMS-0708832 and by ISF grant no. 120/06.

REFERENCES

1. A. Aceves, C. De Angelis, G. Luther, A. Rubenchik, and S. Turitsyn (1995). All-optical-switching and pulse amplification and steering in nonlinear fiber arrays. *Physica D* 87:262–272.
2. A. Aceves, C. De Angelis, A. Rubechik, and S. Turitsyn (1994). Multidimensional solitons in fiber arrays. *Opt. Lett.* 19:329–331.
3. A. Aceves, C. De Angelis, G. Luther, A. Rubenchik, and S. Turitsyn (1995). Energy localization in nonlinear fiber arrays: Collapse effect compressor. *Phys. Rev. Lett.* 75:73–76.
4. A. Aceves, C. De Angelis, G. Luther, A. Rubenchik, and S. Turitsyn (1995). Optical pulse compression using fiber arrays. *Optical Fiber Technol.* 1:244–246.

5. Y. Cao, Z.H. Musslimani, and E.S. Titi (2008). Nonlinear Schrödinger–Helmholtz equation as numerical regularization of the nonlinear Schrödinger equation. *Nonlinearity* 21:879–898.
6. T. Cazenave (2003). *Semilinear Schrödinger Equations*. Courant Lecture notes in Mathematics. AMS, Providence, RI.
7. L.C. Evans (2000). *Partial Differential Equations*. Graduate Studies in Mathematics, Vol. 19. AMS, Providence.
8. G. Fibich (2001). Self-focusing in the damped nonlinear Schrödinger equation. *SIAM J. Appl. Math.* 61:1680–1705.
9. G. Fibich, B. Ilan, and G. Papanicolaou (2002). Self-focusing with fourth-order dispersion. *SIAM J. Appl. Math.* 62:1437–1462.
10. G. Fibich and D. Levy (1998). Self-focusing in the complex Ginzburg–Landau limit of the critical nonlinear Schrödinger equation. *Phys. Lett. A.* 249:286–294.
11. G. Fibich and G. Papanicolaou (1998). A modulation method for self-focusing in the perturbed critical nonlinear Schrödinger equation. *Phys. Lett. A* 239:167–173.
12. G. Fibich and G. Papanicolaou. Self-focusing in the perturbed and unperturbed nonlinear Schrödinger equation in critical dimension. *SIAM J. Appl. Math.* 60:183–240.
13. J. Ginibre and G. Velo (1979). On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *J. Funct. Anal.* 32:1–32.
14. R.T. Glassey (1977). On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation. *J. Math. Phys.* 18:1794–1797.
15. T. Kato (1987). On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 46:113–129.
16. E. Laedke, H. Spatschek, and S. Turitsyn (1995). Analytics criterion for soliton instability in a nonlinear fiber array. *Phys. Rev. E.* 52:5549–5554.
17. M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1988). Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension. *Phys. Rev. A* 38:3837–3843.
18. V. Malkin (1993). On the analytical theory for stationary self-focusing of radiation. *Physica D* 64:251–266.
19. C. Sulem and P.L. Sulem (1999). *The Nonlinear Schödinger Equation, Self-Focusing and Wave Collapse*. Applied Mathematical Sciences, Vol. 139. Springer-Verlag, New York.
20. M.I. Weinstein (1983). Nonlinear Schrödinger equations and sharp interpolation estimates. *Commun. Math. Phys.* 87:567–576.
21. M.I. Weinstein and B. Yeary (1996). Excitation and dynamics of pulses in coupled fiber arrays. *Phys. Lett. A* 222:157–162.