ABSTRACT BIOLOGICAL SYSTEMS AS
SEQUENTIAL MACHINES

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It is shown that a rather close relationship exists between the \((\mathcal{M}, R)\)-systems, defined previously as prototypes of abstract biological systems, and the sequential machines which have been studied by various authors. The theory of sequential machines is reformulated in a way suitable for its application to the study of the intertransformability of \((\mathcal{M}, R)\)-systems as a result of environmental alteration. The important concept of strong connectedness is most useful in this direction, and is used to derive a number of results on intertransformability. Some suggestions are made for further studies along these lines.

1. Introduction. The purpose of this note is to point out certain similarities which exist between the theory of sequential machines, as developed by E. F. Moore (1956), D. A. Huffman (1955), and others [cf. Ginsburg (1962) and Glushkov (1961) for a more extensive bibliography], and the theory of \((\mathcal{M}, R)\)-systems, which we have proposed as a relational prototype of freeliving biological systems (Rosen, 1958a,b, 1959, 1961, 1962, 1963). These two theories have grown up independently, and despite their formal similarities each of them possesses its own problems and its own emphasis. Nevertheless, a number of important concepts are common to both, and when this is pointed out, it will be seen that each of these theories can serve to widen the scope of the other.

In Section 2 below we define the sequential machine and develop a few of its properties that will be useful to us later, although no exhaustive analysis of any of these properties is attempted. Some of this development is a paraphrase of parts of the analysis of Glushkov (loc. cit.). In Section 3 we show how
the notion of the $(\mathcal{M}, \mathcal{R})$-system is related to that of the sequential machine, and show how some of the results developed in Section 2 can be applied to some fundamental problems in the dynamics of $(\mathcal{M}, \mathcal{R})$-systems, which have already been discussed in some detail (Rosen, 1961, 1963).

2. Sequential Machines. A sequential machine (cf. Ginsburg, loc. cit.) is generally defined as a 5-tuple

$$\Lambda = \{S, M, N, \delta, \lambda\}$$

where:

- $S$ is the set of states of the machine $\Lambda$;
- $M$ is the set of inputs to $\Lambda$;
- $N$ is the set of outputs of $\Lambda$;
- $\delta: M \times S \rightarrow S$ is the next state function;
- $\lambda: M \times S \rightarrow N$ is the output function.

In the literature on sequential machines, the set $M$ is sometimes regarded as a collection of elementary input symbols (i.e., as an input alphabet) and sometimes as a set of sequences of such symbols. For our purposes it will be convenient to regard $M$ as an input alphabet rather than as a set of sequences; we shall accordingly do so throughout the sequel.

A sequential machine $\Lambda$ is called strongly connected if, given any pair of states $s_1, s_2 \in S$, there exists a sequence of inputs which carries $\Lambda$ from state $s_1$ to state $s_2$. The strongly connected machines are interesting in themselves, and we shall see below that the concept of strong connectedness has important applications to the theory of $(\mathcal{M}, \mathcal{R})$-systems.

Before we can proceed it is necessary to recast the notion of sequential machine into a (for our purposes) somewhat more perspicuous form. We begin with the elementary observation that the input alphabet $M$ of a sequential machine $\Lambda = \{S, M, N, \delta, \lambda\}$ can be embedded in a 1-1 fashion into the set of mappings $H(S, S)$ of $S$ into itself. This correspondence is established by associating with each $m \in M$ the mapping $\phi_m \in H(S, S)$ defined by

$$\phi_m(s) = \delta(m, s).$$

Conversely: Any subset $A \subset H(S, S)$ defines a state function for a sequential machine whose set of states is $S$.

Precisely the same argument can be used to embed $M$ in a 1-1 fashion into the set $H(S, N)$ via the output map $\lambda$. However, since we shall be restricting attention in the present note to the concept of strong connectedness, which does not involve the output map, we shall not here pursue considerations involving $N$ or $\lambda$.

In fact, as far as strong connectedness is concerned, the sequential machine $\Lambda$ becomes in effect the triple $\{S, M, \delta\}$. What we have shown above is that
this triple can be replaced by the pair \((S, A)\), where \(S\) is a set and \(A\) is an arbitrary subset of \(H(S, S)\). The set \(A\) is in 1–1 correspondence with the elements of the “input alphabet” for the machine; the next state function is defined by indexing the elements of \(A\) with the elements of a set \(M\) (which can then be regarded as the set of input symbols to the machine) and using (1) above.

Henceforth, then, we can consider a general sequential machine as such a pair \((S, A)\).

Since \(A\) is in 1–1 correspondence with the input alphabet of the sequential machine \(\Lambda = \{S, M, \delta\}\), we can represent a sequence \(m_1 m_2(m_1, m_2 \in M)\) of inputs to \(\Lambda\) by the composite mapping \(\phi m_1 \phi m_2\), where \(\phi m_1 \phi m_2\) are the corresponding mappings of \(A\). It is readily verified that sequences of any length, say \(m_1 m_2 \ldots m_r\), are represented by the corresponding composite of mappings \(\phi m_1 \phi m_2 \ldots \phi m_r\) in \(A\).

Now the set of all sequences of elements of \(M\) is the free monoid generated by \(M\). However, if we look at the representation \(\Lambda = (S, A)\), we know that any composite of maps in \(A \subset H(S, S)\) is again an element of \(H(S, S)\). Now \(H(S, S)\), the set of all maps of the set \(S\) into itself, is itself a monoid under the composition operation, which is in general not free; for example, if \(S\) is a finite set containing \(N\) elements, then the cardinality of \(H(S, S)\) is \(N^N\), while the cardinality of any free monoid is infinite. Thus, if we define \(A' \subseteq H(S, S)\) to be the set of all composites of maps in \(A\), then \(A'\) is the submonoid of \(H(S, S)\) which is generated by the elements of \(A\), but which is not free on these generators. That is, there exists a set of defining relations (cf., for example, Clifford and Preston, 1961, p. 40 et seq.) which holds on the set of composites of the mappings in \(A\), and which depends on the structure of the given machine. Just how these relations depend on the given machine is an extremely interesting and rather subtle algebraic problem, which, however, we shall not touch upon here.

Let us see how the concept of strong connectedness is mirrored in the formulation of the theory of sequential machines which has been outlined above. It is immediate from the definitions that a machine \((S, A)\) will be strongly connected if, and only if, for any pair \((s_i, s_j)\) of elements of \(S\), there is a mapping \(\phi \in A'\) such that \(\phi(s_i) = s_j\).

This condition can be stated in a different form. Let us note that any map \(\phi \in H(S, S)\) may be regarded as a subset \(T_\phi\) of the cartesian product \(S \times S\). A family of mappings \(A\) in turn defines a subset \(T_A = \cup_{\phi \in A} T_\phi\). The condition for strong connectedness of a machine \((S, A)\) is thus expressed by the condition that the subset \(T_A'\) of \(S \times S\) shall actually be all of \(S \times S\). The significance of this condition is the following: since \(T_A'\) is all of \(S \times S\), it
follows that any mapping \( f \in H(S, S) \), regarded as a subset of \( S \times S \), is a proper subset of \( T_{A'} \). To speak roughly but graphically, each mapping \( f \in H(S, S) \) can be constructed by pasting together local pieces of mappings in \( A' \) (which, it must be remembered, is in general not all of \( H(S, S) \), although \( T_{A'} \) is all of \( S \times S \)). A moment's reflection will reveal that, in this formulation, we have shown that a strongly connected machine on a set \( S \) of states can realize any function on \( S \) by an appropriate labeling of inputs. Since any machine on \( S \) is a set of such mappings, we can therefore, in a real sense, use a strongly connected machine on \( S \) to imitate the action of any machine on \( S \) whatever.

Let us make the preceding ideas clear by means of an example. Let \( S \) be a set of four elements, \( S = \{s_1, s_2, s_3, s_4\} \). Let the set \( A \) contain the single mapping \( \sigma \), where \( \sigma \) is a cyclic permutation of \( S \) (i.e., \( \sigma(s_1) = s_2, \sigma(s_2) = s_3, \sigma(s_3) = s_4, \sigma(s_4) = s_1 \)). Then \( A' \) consists of the mappings \( \sigma, \sigma^2, \sigma^3 \), and the identity, and it is immediately verified that \( T_{A'} \) is all of \( S \times S \). Now let \( f \in H(S, S) \) be any given map; for definiteness suppose \( f \) is defined as follows:

\[
\begin{align*}
   f(s_1) &= s_3, \quad f(s_2) = s_1, \quad f(s_3) = s_4, \quad f(s_4) = s_1.
\end{align*}
\]

Then we have the equations

\[
\begin{align*}
   f(s_1) &= \sigma^2(s_1), \quad f(s_2) = \sigma^3(s_2), \quad f(s_3) = \sigma(s_3), \quad f(s_4) = \sigma(s_4).
\end{align*}
\]

Thus we see explicitly how we can piece together the function \( f \) from pieces of the mappings in \( A' \).

Retreating back to the customary language of machine theory, the machine \( \Lambda = (S, \{\sigma\}) \) is equivalent to the machine \( (S, \{a\}, \delta) \) where \( a \) is a single alphabet symbol and \( \delta(a, s_i) = \sigma(s_i) \). If another machine \( \Lambda' = (S, \{a'\}, \delta') \) is given, where \( a' \) is another alphabet symbol and \( \delta' \) is defined by \( \delta'(a', s_i) = f(s_i) \), where \( f \) is given as above, then the above discussion translates precisely into the statement that every activity of the machine \( \Lambda' \) can be imitated by supplying appropriate input sequences to the strongly connected machine \( \Lambda \).

The above seems to be the essential significance of the strongly connected machines. It may be mentioned in passing here that the idea of "pasting together" a mapping from local pieces is a common and very important mathematical procedure, and one which we believe will have many significant applications to theoretical biology. The result above, that the strongly connected machines are those which, in a sense, provide enough pieces to paste together everything, may together with the subsequent discussion provide one very small motivation for this belief.

Now let us return briefly to the general machine \( (S, A) \). The question arises: if \( (S, A) \) is not strongly connected, can I enlarge \( A \) so that the resulting machine is strongly connected? The answer is obviously yes; in fact, adjoining the single permutation \( \sigma \) defined above to \( A \) will suffice to make the enlarged machine strongly connected. That is: given any sequential machine \( \Lambda = (S, A) \), there exists a strongly connected machine \( \Lambda_0 = (S, A_0) \) such that \( A \) is a proper
subset of \( A_0 \); i.e., in more standard terminology, \( A \) is obtained from \( A_0 \) by throwing away some input symbols to \( A_0 \).

This result can be generalized to the following: if \( A_1 = (S, A_1) \) and \( A_2 = (S, A_2) \), then there exists a strongly connected machine \( A_0 = (S, A_0) \) such that, speaking roughly, \( A_1 \) and \( A_2 \) can be obtained from \( A_0 \) by throwing away appropriate subsets of \( A_0 \). We can exhibit such a machine \( A_0 \) by writing \( A_0 = A_1 \cup A_2 \cup \{\sigma\} \). This can clearly be generalized to any number of sequential machines with \( S \) as its set of states.

3. Sequential Machines and \((\mathcal{M}, \mathcal{R})\)-Systems. It may be well to begin with a few examples. Let us first consider the simplest \((\mathcal{M}, \mathcal{R})\)-system \( \{f, \Phi_f\} \), where \( f \in H(A, B) \) and \( \Phi_f \in H(B, H(A, B)) \). If we make the following identifications:

\[
\begin{align*}
S &\equiv H(A, B) \\
M &\equiv A \\
N &\equiv B \\
\delta &\colon A \times H(A, B) \to H(A, B) \equiv \delta(a, f) = \Phi_f(f(a)) \\
\lambda &\colon A \times H(A, B) \to B \equiv \lambda(a, f) = f(a)
\end{align*}
\]

we can verify immediately that the 5-tuple \((S, M, N, \delta, \lambda)\) forms a sequential machine.

Next, consider the \((\mathcal{M}, \mathcal{R})\)-system \( \{f, g, \Phi_f, \Phi_g\} \). Here we have \( f \in H(A, B) \), \( g \in H(C, D) \), \( \Phi_f \in H(B \times D, H(A, B)) \), \( \Phi_g \in H(B \times D, H(C, D)) \). If we now make the identifications:

\[
\begin{align*}
S &\equiv H(A, B) \times H(C, D) \\
M &\equiv A \times C \\
N &\equiv B \times D \\
\delta &\colon S \times M \to S \equiv \delta[(f, g), (a, c)] = [\Phi_f(f(a), g(c)), \Phi_g(f(a), g(c))] \\
\lambda &\colon S \times M \to S \equiv \lambda[(f, g), (a, c)] = [f(a), g(c)]
\end{align*}
\]

it is once again immediately verified that the 5-tuple \((S, M, N, \delta, \lambda)\) is again a sequential machine.

From this last example we can see how to proceed in the most general case: let \( \{f_1, f_2, \ldots, f_n; \Phi_{f_1}, \Phi_{f_2}, \ldots, \Phi_{f_n}\} \) be a general \((\mathcal{M}, \mathcal{R})\)-system, where \( f_i \in H(A_i, B_i) \) and \( \Phi_{f_i} \in H(\Pi_i B_i, H(A_i, B_i)) \). We make the identifications

\[
\begin{align*}
S &\equiv \Pi_i H(A_i, B_i) \\
M &\equiv \Pi_i A_i \\
N &\equiv \Pi_i B_i
\end{align*}
\]
\[ \delta: S \times M \to S \] is defined by
\[ \delta((f_1, f_2, \ldots, f_n), (a_1, a_2, \ldots, a_n)) = [\Phi_{f_1}(f_1(a_1), f_2(a_2), \ldots, f_n(a_n)), \ldots, \Phi_{f_n}(f_1(a_1), f_2(a_2), \ldots, f_n(a_n))], \]
\[ \lambda: S \times M \to N \] is defined by
\[ \lambda((f_1, f_2, \ldots, f_n), (a_1, a_2, \ldots, a_n)) = (f_1(a_1), f_2(a_2), \ldots, f_n(a_n)). \]

It should be noted that not all the generality which appears to be implicit in the above identifications actually obtains: for example, in the general \((\mathcal{M}, \mathcal{R})\)-system not every \(B_i\) is an environmental output, so that the \(\Phi_{f_i}\), as defined above, are all factorable through subproducts of their domains (cf. Rosen, 1958b).

The above identifications will be seen to transform the general \((\mathcal{M}, \mathcal{R})\)-system into a 5-tuple which is a sequential machine. However, the converse is not in general true, i.e., not every sequential machine arises from some \((\mathcal{M}, \mathcal{R})\)-system. But the fact that a general \((\mathcal{M}, \mathcal{R})\)-system may itself be regarded as a sequential machine, via the identifications given above, shows that the formalism of sequential machines can be usefully applied to certain aspects of the theory of general \((\mathcal{M}, \mathcal{R})\)-systems.

We have in previous work treated in some detail the problem of the reversibility of environmentally induced alterations (Rosen 1961, 1963). We have pointed out in this work that (a) the problem of reversibility, and the more general problem of the intertransformability of abstract biological systems, were among the most important open questions facing any theoretical biology, and (b) that the solution to these questions within the context of the \((\mathcal{M}, \mathcal{R})\)-systems depended in any case on how “rich” the base category is in mappings between the objects in the category. It will be seen that the relationship between the \((\mathcal{M}, \mathcal{R})\)-systems and in general sequential machines may help throw further light on these questions.

Let us consider first the simplest \((\mathcal{M}, \mathcal{R})\)-system, \(\{f, \Phi_f\}\), which we have seen is equivalent to a sequential machine \(\{H(A, B), A, B, \delta, \lambda\}\) via the identification described above. We may note explicitly that here \(H(A, B)\) is a set of mappings in the category of discourse from which our systems are formed, and thus need not consist, \textit{a priori}, of the full set of possible set-theoretic mappings of \(A\) into \(B\). It will be immediately verified that what we have called an environmentally induced alteration in the (“metabolic”) structure of such an \((\mathcal{M}, \mathcal{R})\)-system (Rosen, 1961, 1963) becomes, in the terminology of sequential machines, a change of state of the associated sequential machine. It will further be recognized that the general question of the intertransformability of two \((\mathcal{M}, \mathcal{R})\)-systems \(\{f, \Phi_f\}, \{g, \Phi_g\}\) with differing “metabolic” structure but
identical "genetic" structure [where $f, g \in H(A, B)$] is equivalent to finding a (sequence of) inputs to the associated machine which carries the state $f$ of that machine onto the state $g$. That is: an $(\mathcal{M}, \mathcal{R})$-system $\{f, \Phi_f\}$ will be transformable, via appropriate environmental alterations, to any other system $\{g, \Phi_g\}$, with the same "genetic" structure but differing "metabolic" structure if, and only if, the associated sequential machine is strongly connected.

If the given system $\{f, \Phi_f\}$ does not give rise to a strongly connected machine, then no sequence of environmental alterations will effect certain changes of state (cf. Rosen, 1961, 1963, where conditions for transformability were discussed in some detail). However, from the results of the preceding section, we know that any sequential machine can be transformed, at least formally, into a strongly connected machine by enlarging the input alphabet in an appropriate manner, and finding suitable extensions of the mappings $\delta, \lambda$. In the context of the theory of categories, we can formulate this kind of enlargement in two different ways, corresponding to the two different ways of looking at sequential machines which were described in the preceding section. Most surprisingly, although these two different methods of enlargement are equivalent for the general theory of sequential machines, they turn out not to be so for the theory of $(\mathcal{M}, \mathcal{R})$-systems.

The first possibility is to attempt to embed the input set $A$ into a larger set $A_0$, and to find a mapping $f_0: A_0 \rightarrow B$ which renders the diagram

\[
\begin{array}{c}
A_0 \\
\downarrow i \quad \downarrow f_0 \\
A & \xrightarrow{f} & B
\end{array}
\]

commutative (here $i$ is the inclusion map). We will also need a mapping $\Phi_{f_0}$ which extends $\Phi_f$ in an analogous manner. The existence of these extensions, which do embed our given $(\mathcal{M}, \mathcal{R})$-system into a new system with a strongly connected sequential machine, depends as usual on the "richness" of the category of discourse. The point to notice, however, is that in the theory of $(\mathcal{M}, \mathcal{R})$-systems, whenever we extend the input alphabet, we also extend the set of states of the associated sequential machine. This is because, by embedding the alphabet $A$ in a larger set $A_0$, we must simultaneously embed the set of states $H(A, B)$ into the larger set $H(A_0, B)$ [or, more accurately, map the larger set down into $H(A, B)$]. This phenomenon does not occur in the theory of sequential machines, where it is generally possible to extend the input alphabet without enlarging the set of states; that we cannot do this directly in the
theory of \((\mathcal{M}, \mathcal{R})\)-systems points to a fundamental difference between the two theories.

Alternatively, we can use the formulation of sequential machine theory developed in the preceding section, in which the input alphabet itself is represented by a set of mappings. In the situation of present interest, the input alphabet \(A\) is embedded into the set \(H[H(A, B), H(A, B)]\) via the correspondence \(a \rightarrow \Phi_f(a)\). The enlargement of a given sequential machine to a strongly connected machine, in this formalism, is obtained by adding appropriate further mappings from \(H[H(A, B), H(A, B)]\) to those obtained by embedding \(A\). Once again, the existence of these mappings within our category is dependent on the “richness” of the category, but in this formalism we do not have to solve an extension problem, and we do not have to enlarge our sets of states to construct our strongly connected machine.

In any event, if an extension to a strongly connected machine, by either of the above methods, is possible within our category, then any state transitions which were impossible in our given system may be carried out in the augmented system (at least formally) and we can drop back to the given system by throwing away the now superfluous input symbols, or mappings, which were adjoined to provide the necessary strong connectedness. The existence of the required new symbols, or mappings, is as we have seen, directly related to the “richness” of the category of discourse.

We have seen above how the theory of sequential machines adds new insights to some of the difficult problems of intertransformability of \((\mathcal{M}, \mathcal{R})\)-systems. We shall now briefly mention how these same problems may enlarge the theory of sequential machines in a useful way. We have considered thus far only those transitions of \((\mathcal{M}, \mathcal{R})\)-systems in which the “metabolic” structure alone was altered, while the “genetic” structure remained unaffected. This is of course the most elementary case (cf. Rosen, 1963, p. 46 et seq.) and is the one to which the theory of sequential machines as it now stands can be directly adapted. If we allow transitions of genetic structure in our \((\mathcal{M}, \mathcal{R})\)-systems, then according to our definitions we will be causing changes in the next-state function \(\delta\) of the associated sequential machine. As far as the author is able to determine, there does not exist any theory of sequential machines with variable state functions. Yet such a theory would be quite timely; aside from the theory of \((\mathcal{M}, \mathcal{R})\)-systems, many other kinds of adaptive devices used in prototypes of cognitive, self-organizing or pattern-recognizing devices (such as the perceptron) are basically of this type. The problems arising from the various adaptive devices mentioned above seem to be quite specialized and do not appear to offer much insight into how a general theory of variable state functions should be developed. The theory of \((\mathcal{M}, \mathcal{R})\)-
systems, on the other hand, by its very generality, may hopefully serve to enable us to fill this gap, with the subsequent enrichment of many fields.

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LITERATURE


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