Final Exam

Math 21B: Section 001

Summer 2012

Name ____________________________

ID number ________________________

Directions

- Do not begin until instructed to do so. You will have 90 minutes to complete the exam.
- You may use pencils, pens, and erasers.
- Put away all books, notes, cell phones, calculators, and other electronic devices.
- Show all work for full credit. If in doubt, write it out.
- Keep your work as neat as possible. If we can't read it, we won't grade it!
- Keep your student ID out on your desk while working on your exam.
- You must also present your student ID to the instructor when turning in your exam.

Point totals

Problem	1	2	3	4	5	6	7	8	9	10	11	total
Score	/8	/8	/8	/12	/8	/8	/12	/12	/16	/12	/16	/120

1. (8 Points) Calculate the indefinite integral

$$\int e^{\theta} \sin(\theta)\, d\theta.$$

$u = \sin\theta \quad dv = e^{\theta} d\theta$

$du = \cos\theta\, d\theta \quad v = e^{\theta}$

$u = \cos\theta \quad dv = e^{\theta} d\theta$

$du = -\sin\theta\, d\theta \quad v = e^{\theta}$

$$\int e^{\theta}\sin\theta\, d\theta = e^{\theta}\sin\theta - \int e^{\theta}\cos\theta\, d\theta$$

$$= e^{\theta}\sin\theta - \left[e^{\theta}\cos\theta - \int -e^{\theta}\sin\theta\, d\theta\right]$$

$$= e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta\, d\theta$$

$$\Rightarrow 2\int e^{\theta}\sin\theta\, d\theta = e^{\theta}\sin\theta - e^{\theta}\cos\theta$$

$$\Rightarrow \int e^{\theta}\sin\theta\, d\theta = \boxed{\frac{e^{\theta}\sin\theta - e^{\theta}\cos\theta}{2} + C}$$

2. (8 Points) Calculate the definite integral

$$\int_1^2 \frac{1}{x^2+3x+2}\, dx.$$

$x^2+3x+2 = (x+1)(x+2)$

$$\Rightarrow \frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Clear denominator

$1 = A(x+2) + B(x+1)$

let $x = -1 \Rightarrow 1 = A$

let $x = -2 \Rightarrow 1 = -B \Rightarrow B = -1$

$$= \int_1^2 \left[\frac{1}{x+1} - \frac{1}{x+2}\right] dx$$

$$= \left(\ln|x+1| - \ln|x+2|\right)\Big]_1^2$$

$$= (\ln|3| - \ln|4|) - (\ln|2| - \ln|3|)$$

$$= 2\ln|3| - 2\ln|2| - \ln|2|$$

$$= \ln|9| - \ln|8|$$

$$\boxed{= \ln\left(\frac{9}{8}\right)}$$

3. (8 points) Please fill in the blanks below to state the **Fundamental Theorem of Calculus.**

If $f(x)$ is continuous on the interval $[a, b]$ and we define the function

$$F(x) = \underline{\int_a^x f(t)dt},$$

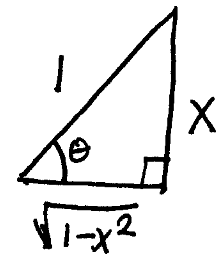
then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b), and has derivative

$$\frac{dF}{dx} = F'(x) = \underline{f(x)}.$$

4. (12 Points) Calculate the indefinite integral

$$\int \frac{x^4}{(1-x^2)^{7/2}}\,dx.$$

NOTE: You may assume that $-1 < x < 1$ for simplicity.



$$\sin\theta = x$$
$$\cos\theta = \sqrt{1-x^2}$$
$$\cos\theta\, d\theta = dx$$

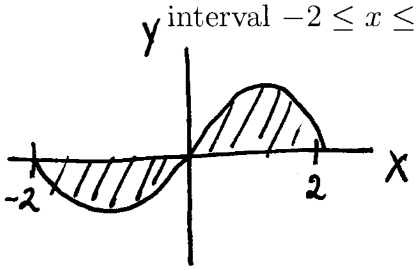
$$\int \frac{x^4}{(1-x^2)^{7/2}}\,dx = \int \frac{\sin^4\theta}{\cos^7\theta}\cos\theta\, d\theta$$
$$= \int \frac{\sin^4\theta}{\cos^6\theta}\,d\theta$$
$$= \int \frac{\sin^4\theta}{\cos^4\theta}\frac{1}{\cos^2\theta}\,d\theta$$
$$= \int \tan^4\theta\sec^2\theta\, d\theta \qquad u=\tan\theta, \quad du=\sec^2\theta\, d\theta$$
$$= \int u^4\,du$$
$$= \frac{u^5}{5}+C$$
$$= \frac{1}{5}\tan^5\theta + C = \boxed{\frac{1}{5}\left(\frac{x}{\sqrt{1-x^2}}\right)^5 + C}$$

5. (8 Points) Find the average value of the function $f(x) = \sin^2(x)$ over the interval $0 \le x \le 5\pi$.

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\hat{f} = \frac{1}{5\pi}\int_0^{5\pi} \sin^2(x)\,dx$$
$$= \frac{1}{5\pi}\int_0^{5\pi}\left(\frac{1-\cos(2x)}{2}\right)dx$$
$$= \frac{1}{10\pi}\left(x - \frac{\sin(2x)}{2}\right)\Bigg]_0^{5\pi}$$
$$= \frac{1}{10\pi}\left[\left(5\pi - \frac{\sin(10\pi)}{2}\right) - \left(0 - \frac{\sin(0)}{2}\right)\right]$$
$$= \frac{1}{10\pi}(5\pi - 0 - 0 + 0)$$
$$\boxed{= 1/2}$$

6. (8 Points) Find the *total area* between the curve $y = x\sqrt{4-x^2}$ and the x-axis on the interval $-2 \le x \le 2$.



Y is odd function of X

$$A = \int_{-2}^{2} \left| x\sqrt{4-x^2} \right| dx = 2\int_0^2 x\sqrt{4-x^2}\, dx$$

$u = 4 - x^2$
$du = -2x\,dx$
$x = 0 \Rightarrow u = 4$
$x = 2 \Rightarrow u = 0$

$$= 2\left(-\tfrac{1}{2}\right)\int_4^0 \sqrt{u}\, du$$

$$= \int_0^4 \sqrt{u}\, du = \tfrac{2}{3}u^{3/2}\Big]_0^4 = \tfrac{2}{3}(4)^{3/2} = \boxed{\tfrac{16}{3}}$$

7. (12 Points) Recall that we defined the function

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$

Use this to show that $\ln(1/x) = -\ln(x)$. For simplicity, you may assume that $x > 0$.

$$\ln\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{1}{t}\, dt$$

let $u = x \cdot t$
$du = x\, dt$
when $t = 1/x \Rightarrow u = 1$
$t = 1 \Rightarrow u = x$

$$= \int_x^1 \frac{1}{u/x}\,\frac{1}{x}\, du$$

$$= \int_x^1 \frac{x}{u}\,\frac{1}{x}\, du$$

$$= \int_x^1 \frac{1}{u}\, du$$

$$= -\int_1^x \frac{1}{u}\, du = -\ln(x)$$

8. (12 Points) Evaluate this improper integral, or show that it does not converge.

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx.$$

$$\int_{-\infty}^{0} 2xe^{-x^2}dx = \lim_{b\to-\infty}\int_{b}^{0} 2xe^{-x^2}dx$$
$$= \lim_{b\to-\infty} -e^{-x^2}\Big]_b^0$$
$$= \lim_{b\to-\infty}(-1+e^{-b^2})$$
$$= -1$$

$$\int_{0}^{\infty} 2xe^{-x^2}dx = \lim_{c\to\infty}\int_{0}^{c} 2xe^{-x^2}dx$$
$$= \lim_{c\to\infty} -e^{-x^2}\Big]_0^c$$
$$= \lim_{c\to\infty}(-e^{-c^2}+1)$$
$$= 1$$

$$\int_{-\infty}^{\infty} 2xe^{-x^2}dx = \int_{-\infty}^{0} 2xe^{-x^2}dx + \int_{0}^{\infty} 2xe^{-x^2}dx = -1+1 = \boxed{0}$$

9. (16 Points) Evaluate this improper integral, or show that it does not converge.

$$\int_0^1 \ln(x)\,dx.$$

Consider $\int_b^1 \ln(x)dx$ let $u=\ln(x)$ $dv=dx$
$du=\frac{1}{x}dx$ $v=x$

$$= \ln(x)\cdot x\Big]_b^1 - \int_b^1 \frac{1}{x}\cdot x\,dx$$
$$= (\ln(x)\cdot x - x)\Big]_b^1$$
$$= \underbrace{\ln(1)}_{0} - 1 - b\ln(b) + b$$

$$\lim_{b\to0} b = 0$$

$$\lim_{b\to0} b\ln(b) = \lim_{b\to0}\frac{\ln(b)}{1/b}$$
L'hospital's Rule
$$= \lim_{b\to0}\frac{1/b}{-1/b^2}$$
$$= \lim_{b\to0} -b$$
$$= 0$$

$$\Rightarrow \int_0^1 \ln(x)dx = \lim_{b\to0}\int_b^1 \ln(x)dx = \lim_{b\to0}(-1+b-b\ln(b)) = \boxed{-1}$$

10. (12 Points)

(a) (6 Points) Use integration by parts to show that

$$\int e^{\sqrt{x}}\,dx = 2e^{\sqrt{x}}\left(\sqrt{x}-1\right)+C.$$

$$\int e^{x^{1/2}}dx = \int x^{1/2}x^{-1/2}e^{x^{1/2}}dx \qquad \begin{cases} u = x^{1/2} \\ du = \frac{1}{2}x^{-1/2}dx \end{cases} \qquad \begin{aligned} dv &= x^{-1/2}e^{x^{1/2}}dx \\ v &= 2e^{x^{1/2}} \end{aligned}$$

$$= 2x^{1/2}e^{x^{1/2}} - \int \tfrac{1}{2}x^{-1/2}\cdot 2e^{x^{1/2}}dx \qquad \begin{cases} u = x^{1/2} \\ du = \frac{1}{2}x^{-1/2} \end{cases}$$

$$= 2x^{1/2}e^{x^{1/2}} - 2e^{x^{1/2}} + C$$

$$= 2e^{x^{1/2}}(x^{1/2}-1) + C$$

(b) (6 Points) Use part (a) to calculate the indefinite integral

$$\int \sqrt{x}e^{\sqrt{x}}\,dx.$$

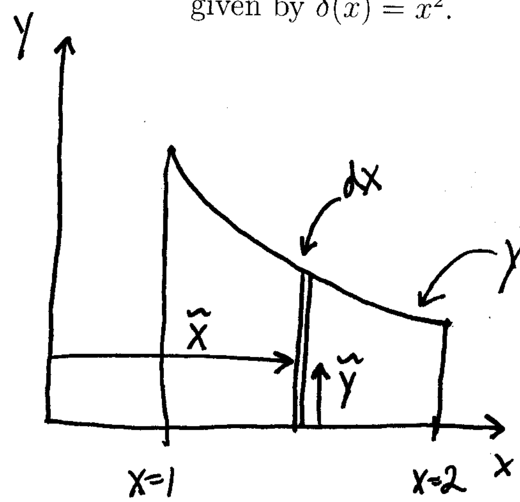
$$\int x^{1/2}e^{x^{1/2}}dx = \int x\,x^{-1/2}e^{x^{1/2}}dx \qquad \begin{aligned} u &= x \\ du &= dx \end{aligned} \qquad \begin{aligned} dv &= x^{-1/2}e^{x^{1/2}}dx \\ v &= 2e^{x^{1/2}} \end{aligned}$$

$$= 2xe^{x^{1/2}} - \int 2e^{x^{1/2}}dx$$

$$= 2xe^{x^{1/2}} - 4e^{x^{1/2}}(x^{1/2}-1) + C$$

$$= 2e^{x^{1/2}}(x - 2x^{1/2} + 2) + C$$

11. (16 Points) Find the center of mass of the thin plate covering the region between the x-axis, the curve $y = \frac{2}{x^2}$, the line $x = 1$ and the line $x = 2$, if the plate has a density given by $\delta(x) = x^2$.



$$dA = \frac{2}{x^2}dx$$

$$dM = \delta\, dA = x^2 \frac{2}{x^2}dx = 2dx$$

$$\tilde{x} = x$$

$$\tilde{y} = \frac{1}{2}\left(\frac{2}{x^2}\right) = \frac{1}{x^2}$$

$$M_y = \int_1^2 \tilde{x}\, dM = \int_1^2 2x\,dx = x^2\Big]_1^2 = 4-1 = 3$$

$$M_x = \int_1^2 \tilde{y}\, dM = \int_1^2 \frac{2}{x^2}dx = \frac{-2}{x}\Big]_1^2 = \frac{-2}{2} + \frac{2}{1} = 1$$

$$M = \int_1^2 dM = \int_1^2 2\,dx = 2x\Big]_1^2 = 4-2 = 2$$

Center of Mass

$$(x, y) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{3}{2}, \frac{1}{2}\right)}$$