Midterm Exam

Math 21B: Section 001

Summer 2012

Name ______________________________

ID number ___________________________

Directions

- Do not begin until instructed to do so. You will have 90 minutes to complete the exam.
- You may use pencils, pens, and erasers.
- Put away all books, notes, cell phones, calculators, and other electronic devices.
- Show all work for full credit. If in doubt, write it out.
- Keep your work as neat as possible. If we can't read it, we won't grade it!
- Keep your student ID out on your desk while working on your exam.
- You must also present your student ID to the instructor when turning in your exam.

Point totals

Problem	1	2	3	4	5	6	7	8	9	10	11	total
Score	/8	/12	/8	/8	/12	/16	/8	/12	/16	/16	/16	/132

1. (8 Points) Solve the following initial value problem:

$$\frac{ds}{dt} = \cos(t) + \sin(t), \quad s(\pi) = 1.$$

Solution: We know that $s(t)$ is an antiderivative of the given function. Therefore we calculate

$$s(t) = \int (\cos(t) + \sin(t))\ dt = \sin(t) - \cos(t) + C.$$

We now use the given initial condition to solve for the unknown constant C.

$$1 = s(\pi) = \sin(\pi) - \cos(\pi) + C = 1 + C, \Rightarrow C = 0.$$

This fully determines the function $s(t) = \sin(t) - \cos(t)$.

2. (12 Points) Suppose that f and g are both integrable functions. You are given that

$$\int_1^3 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 1, \quad \int_{-1}^3 g(x)\,dx = 7/2, \quad \int_{-1}^5 g(x)\,dx = 1/2.$$

Use these facts to calculate the definite integral

$$\int_5^3 (f(x) - 2g(x))\ dx.$$

Solution: We use the properties of the definite integral

$$\begin{aligned}
\int_5^3 (f(x) - 2g(x))\ dx &= -\int_3^5 (f(x) - 2g(x))\ dx \\
&= \int_3^5 (2g(x) - f(x))\ dx \\
&= 2\int_3^5 g(x)\,dx - \int_3^5 f(x)\,dx \\
&= 2\left(\int_{-1}^5 g(x)\,dx - \int_{-1}^3 g(x)\,dx\right) - \left(\int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx\right) \\
&= 2\,(1/2 - 7/2) - (1 - -4) = -11.
\end{aligned}$$

3. (8 Points) Calculate the definite integral

$$\int_0^1 xe^{x^2}\,dx.$$

Solution: We let $u = x^2$. A quick calculation shows $du = 2xdx$ and thus $xdx = \frac{1}{2}du$. When $x = 0$ we have $u = 0$, and similarly when $x = 1$, $u = 1$. We use these facts to make the substitution

$$\begin{aligned} \int_0^1 xe^{x^2}\,dx &= \int_0^1 \frac{1}{2}e^u\,du \\ &= \frac{1}{2}e^u\Big]_0^1 \\ &= \frac{1}{2}\left(e^1 - e^0\right) \\ &= \frac{e-1}{2} \end{aligned}$$

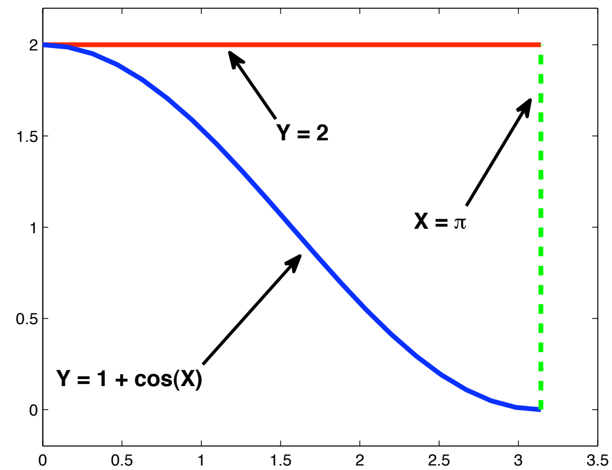
4. (8 Points) Find the indefinite integral

$$\int \frac{t\sqrt[3]{t} - \sqrt[3]{t^2}}{t^2}\,dt$$

Solution:

$$\begin{aligned} \int \frac{t\sqrt[3]{t} - \sqrt[3]{t^2}}{t^2}\,dt &= \int \frac{t \cdot t^{1/3} - (t^2)^{1/3}}{t^2}\,dt \\ &= \int \frac{t^{4/3} - t^{2/3}}{t^2}\,dt \\ &= \int t^{-2/3} - t^{-4/3}\,dt \\ &= 3t^{1/3} + 3t^{-1/3} + C. \end{aligned}$$

5. (12 Points) Sketch the region between the curves $y = 1 + \cos(x)$, $y = 2$, and $x = \pi$. Then calculate the area of this region. Solution: The region in question is pictured in the figure below.



We calculate its area using the formula $A = \int_a^b h(x)\,dx$. The function $h(x)$ is the height of the region of interest, which in this case is from the top curve ($y = 2$) to the bottom curve ($y = 1 + \cos(x)$). Therefore

$$h(x) = 2 - (1 + \cos(x)) = 1 - \cos(x).$$

As can be seen in the figure, the region spans from $x = 0$ to $x = \pi$. Thefore

$$\begin{aligned} A &= \int_0^{\pi} (1 - \cos(x))\ dx \\ &= x - \sin(x)]_0^{\pi} \\ &= (\pi - \sin(\pi)) - (0 - \sin(0)) \\ &= \pi \end{aligned}$$

6. (16 Points) Solve the following initial value problem:

$$\frac{d^2y}{d\theta^2} = 4\sec^2(2\theta)\tan(2\theta), \quad y'(0) = 4, \quad y(0) = -1.$$

Solution: As before, we begin by finding the general antiderivative. To do this we make the substitution $u = \sec(2\theta)$. A quick calculation shows that $du = 2\sec(2\theta)\tan(2\theta)d\theta$. Therefore

$$\int 4\sec^2(2\theta)\tan(2\theta)\,d\theta = \int 2u\,du = u^2 + C = \sec^2(2\theta) + C.$$

We use the initial condition for $y'(\theta)$ to solve for the unknown constant

$$4 = y'(0) = \sec^2(0) + C \Rightarrow C = 3.$$

This gives us that $\frac{dy}{d\theta} = \sec^2(2\theta) + 3$. We repeat this process to find $y(\theta)$. Taking the antiderivative we have

$$\int \frac{dy}{d\theta}\,d\theta = \int \left(\sec^2(2\theta) + 3\right)\,d\theta = \frac{1}{2}\tan(2\theta) + 3\theta + C.$$

Using our initial condition we see

$$-1 = y(0) = \frac{1}{2}\tan(0) + 3(0) + C \Rightarrow C = -1.$$

Therefore

$$y(\theta) = \frac{1}{2}\tan(2\theta) + 3\theta - 1.$$

7. (8 Points) Find the average value of the function $f(x) = \sin(x)$ over the interval $[0, 7\pi]$.
Solution: We know that the average value of a function over an interval is given by $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$. Therefore we calculate

$$\begin{aligned}\bar{f} &= \frac{1}{7\pi - 0}\int_0^{7\pi} \sin(x)\,dx \\ &= \frac{-1}{7\pi}\cos(x)\Big]_0^{7\pi} \\ &= \frac{-1}{7\pi}\left(\cos(7\pi) - \cos(0)\right) \\ &= \frac{2}{7\pi}\end{aligned}$$

8. (12 Points) Find the length of the curve given by $y = \frac{2}{3}(x^2+1)^{3/2}$ on the interval $0 \le x \le 2$.

Solution: We begin by recalling the formula for arc length:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

In order to use this formula, we calculate

$$\frac{dy}{dx} = 2x(x^2+1)^{1/2}.$$

$$\left(\frac{dy}{dx}\right)^2 = 4x^2(x^2+1) = 4x^4 + 4x^2.$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 4x^4 + 4x^2 + 1 = (2x^2+1)^2,$$

and therefore

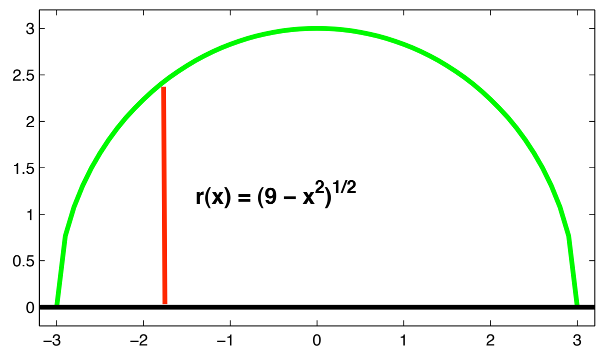
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2x^2 + 1.$$

This allows us to use the arc length formula and calculate

$$\begin{aligned} L &= \int_0^2 \left(2x^2+1\right)\, dx \\ &= \frac{2}{3}x^3 + x\Big]_0^2 \\ &= \frac{2}{3}(2^3) + 2 \\ &= \frac{22}{3} \end{aligned}$$

9. (16 Points) Calculate the volume of a sphere of radius 3 by rotating a semi-circular arc (of radius 3) about the x-axis (HINT: The equation for a circle of radius r in the x-y plane is $y^2 + x^2 = r^2$).

Solution: We begin by drawing the semi-circle (or radius 3) in the top half plane. We define the distance from the x-axis to this curve to be $r(x)$.

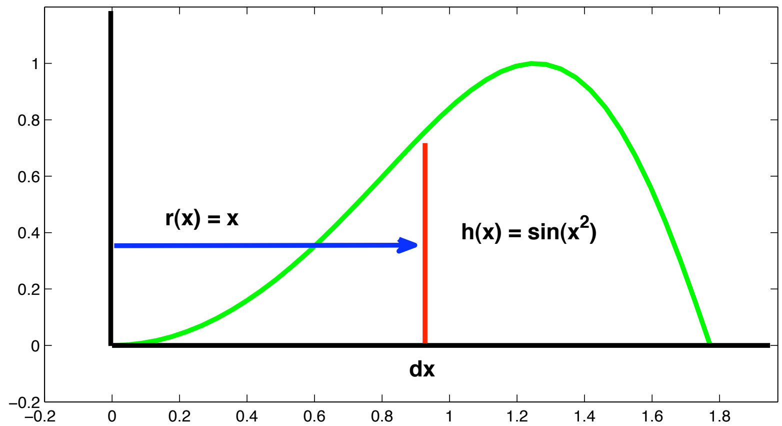


The formula given in the hint allows us to solve for $r(x) = y = \sqrt{9 - x^2}$. Using the disk method, we know that rotating this semi-circle about the x-axis will produce a volume $V = \int_a^b \pi r(x)^2 \, dx$. This is the sphere we are interested in. Our limits of integration are where the curve touches the x-axis, which is $x = -3$ and $x = 3$. Using this information, we calculate

$$\begin{aligned} V &= \int_{-3}^{3} \pi \left(\sqrt{9 - x^2}\right)^2 \, dx \\ &= \int_{-3}^{3} \pi (9 - x^2) \, dx \\ &= \pi (9x - \frac{x^3}{3})\Big]_{-3}^{3} \\ &= \pi \left(27 - \frac{27}{3}\right) - \pi \left(-27 + \frac{27}{3}\right) \\ &= \pi(27 - 9 + 27 - 9) \\ &= 36\pi \end{aligned}$$

10. (16 Points) Calculate the volume of the solid that is generated when you take the region bounded by the x-axis and the graph of $f(x) = \sin(x^2)$ (on the interval $0 \leq x \leq \sqrt{\pi}$) and revolve it about the y-axis.

Solution: We will calculate this volume using the method of cylindrical shells. Begin by sketching the region being rotated and drawing a slice through it parallel to the axis of revolution.



This slice has a thickness that we call dx. It is a distance $r(x) = x$ from the axis of revolution. It stretches from the x-axis to the curve $y = \sin(x^2)$. Therefore the height of this slice is $h(x) = \sin(x^2)$. The method of cylindrical shells tells us that the volume of the solid of revolution is

$$V = \int_a^b 2\pi r(x)h(x)\,dx = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\,dx.$$

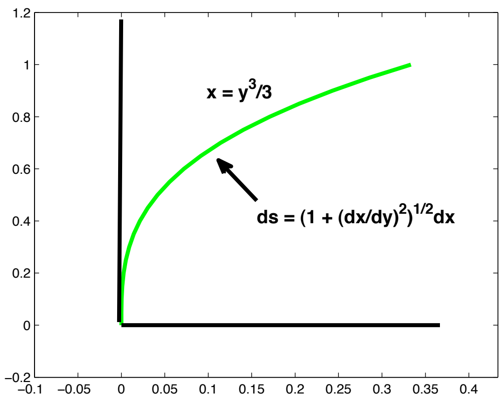
To evaluate this integral, we make the substitution $u = x^2$. A quick calculation shows us that $du = 2xdx$, when $x = 0$, $u = 0$, and when $x = \sqrt{pi}$, $u = \pi$. Using this information, we make a U-substitution and get

$$V = \int_0^{\pi} \pi \sin(u)\,du = -\pi\,\cos(u)]_0^{\pi} = 2\pi.$$

11. (16 Points) Calculate the lateral surface area (*not* including the circular end) of the solid that is generated when you take the graph of the function $g(y) = y^3/3$ on the interval $0 \leq y \leq 1$ and revolve it about the y-axis.

Solution: The figure below shows the curve that will be rotated about the y-axis. Recall that the area for surfaces of revolution is

$$A = \int_a^b 2\pi x(y)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$



We begin by calculating the terms needed to evaluate this integral:

$$\frac{dx}{dy} = y^2, \quad \left(\frac{dx}{dy}\right)^2 = y^4, \quad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{y^4 + 1}.$$

The problem statement tells us that we will be integrating from $y = 0$ to $y = 1$. Therefore we must evaluate the integral

$$A = \int_0^1 \frac{2\pi}{3} y^3 \sqrt{y^4 + 1}\, dy.$$

To do this, we make the substitution that $u = y^4 + 1$. A calculation shows us that $du = 4y^3 dy$, and therefore $\frac{2}{3}y^3 dy = \frac{1}{6}du$. Furthermore, when $y = 0$, we have $u = 1$, and when $y = 1$, $u = 2$. This allows us to make the substitution

$$\begin{aligned} A &= \int_0^1 \frac{2\pi}{3} y^3 \sqrt{y^4 + 1}\, dy \\ &= \int_1^2 \frac{\pi}{6}\sqrt{u}\, du \\ &= \frac{\pi}{9}\, u^{3/2}\Big]_1^2 \\ &= \frac{\pi}{9}\left(2^{3/2} - 1^{3/2}\right) \\ &= \frac{\pi}{9}\left(2\sqrt{2} - 1\right) \end{aligned}$$