Owen Lewis MAT 21B/Summer Session 2012

Name: ______________________________ Student ID: ________________

Quiz 3

Directions: You will have 30 minutes to complete this quiz. Please show all of your work and mark your answers clearly. You may not use any extra resources during the quiz: not your notes, not your book, not a cell phone, not a calculator. Good luck.

1. (4 points) If a force of 90 N stretches a spring 1 m beyond its natural length, how much work is required to stretch the same spring 5 m beyond its natural length? HINT: Your answer should have units of Joules (denoted by a capital J).

   Solution: First, we find the spring constant k for Hooke's law. We know that $F = kx$, where F is the force applied and x is the distance the spring is stretched. Substituting in the numbers given in the problem statement we see

   $$90\text{ N} = k \cdot 1\text{ m}.$$

   Solving this for k gives $k = 90\frac{\text{N}}{\text{m}}$. Now that we know the spring constant, we can calculate the work done to stretch the spring.

   Work is defined as $W = \int_a^b F(x)\,dx$. Therefore

   $$W = \int_0^5 kx\,dx = \int_0^5 90x\,dx = 45x^2\Big]_0^5 = 45 \cdot 25 = 1125\text{ J (or N}\cdot\text{m).}$$

2. (6 points) A thin plate covers the region bounded below by the parabola $y = x^2$ and above by the line $y = x$. The plate's density at a point (x, y) is given by $\delta(x) = 10x$. Find M_y, the moment of the plate about the y-axis (HINT: We defined $M_y = \int x\,dM$).

   Solution: While reading this solution, please reference the figure on the last page.

   We cut the plate into thin strips of thickness dx. Each of these strips has thickness $h(x) = x - x^2$. Therefore each has an area $dA = h(x)dx = (x-x^2)dx$. To find the mass of each strip, we multiply its area by the density function $\delta(x)$. Therefore $dM = 10x(x - x^2)dx$. To find the limits of integration, we set the top and bottom curves equal to each other, then solve for x.

   $$x = x^2 \Rightarrow x = 1 \text{ and } x = 0.$$

   Finally, we multiply the moment arm to each strip (x), by the mass dM and integrate

   $$\begin{aligned}
   M_y &= \int_0^1 x\,dM \\
   &= \int_0^1 10 \cdot x \cdot x(x - x^2)\,dx \\
   &= 10\int_0^1 (x^3 - x^4)\,dx \\
   &= 10\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big]_0^1 \\
   &= 10\left(\frac{1}{4} - \frac{1}{5}\right) \\
   &= \frac{1}{2}
   \end{aligned}$$

3. (4 points) Using the definition of a^x discussed in class, calculate the derivative of the function

$$f(x) = a^x \quad \text{where } a > 0.$$

HINT: You may only assume that $\frac{d}{dx}e^x = e^x$. Otherwise, show all of your work. Simply writing down the answer will get you no credit.

Solution: As discussed in class, the we defined exponentiation as

$$a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)}.$$

Using this definition, we calculate

$$\begin{aligned}
\frac{d}{dx}\left[a^x\right] &= \frac{d}{dx}\left[e^{x\ln(a)}\right] \\
&= e^{x\ln(a)} \cdot \frac{d}{dx}\left[x\ln(a)\right] \\
&= e^{x\ln(a)} \cdot \ln(a) \\
&= \ln(a)a^x
\end{aligned}$$

4. (6 points) A cylindrical cistern is 20 ft tall and has a radius of 10 ft. The top of the cistern is burried 10 ft below ground level. If it is filled with a fluid of weight density w, how much work is required to pump all the fluid up to ground level.

Solution: While reading this solution, please reference the figure on the last page.

First, we define the y-axis to originate at ground level, and proceed downward. This means that the top of the cistern is at $y = 10$ and the bottom of the cistern is at $y = 30$. There are other ways to define the axis, but this way will lead to the simplest calculation. We cut the tank into horizontal slices (or cross-sections) perpendicular to the y-axis. Each cross-section has a thickness dy. Each cross-section has a volume $dV = A(y)dy$. To find a formula for $A(y)$, we note that each cross section is a circle of radius 10 ft. Therefore, $A(y) = \pi r^2 = 100\pi$. The weight of fluid in each cross section (and therefore the force required to lift it) is the fluid weight density times the volume: $wdV = 100\pi wdy$. Finally, we calculate the distance that each cross-section of fluid must be pumped. The cross-section must be pumped from its initial height y to the origin of our axis, for a total distance of $D = y$. So, we calculate the total work required as

$$\begin{aligned}
W &= \int_a^b wDdV \\
&= \int_{10}^{30} 100\pi wy\,dy \\
&= 50\pi w \int_{10}^{30} 2y\,dy \\
&= 50\pi w\; y^2\Big]_{10}^{30} \\
&= 50\pi w(900 - 100) \\
&= 40000\pi w
\end{aligned}$$

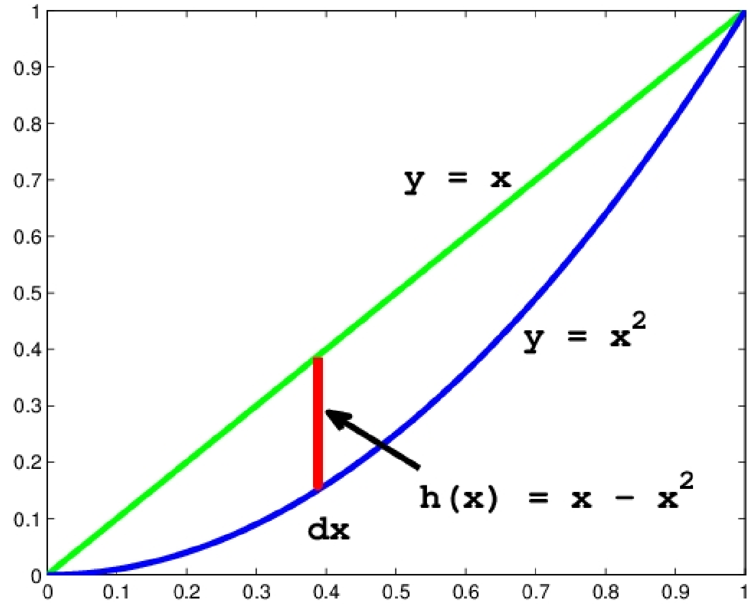

Figure 1: The region of interest for Problem 2. A differential slice of thickness dx is shown.

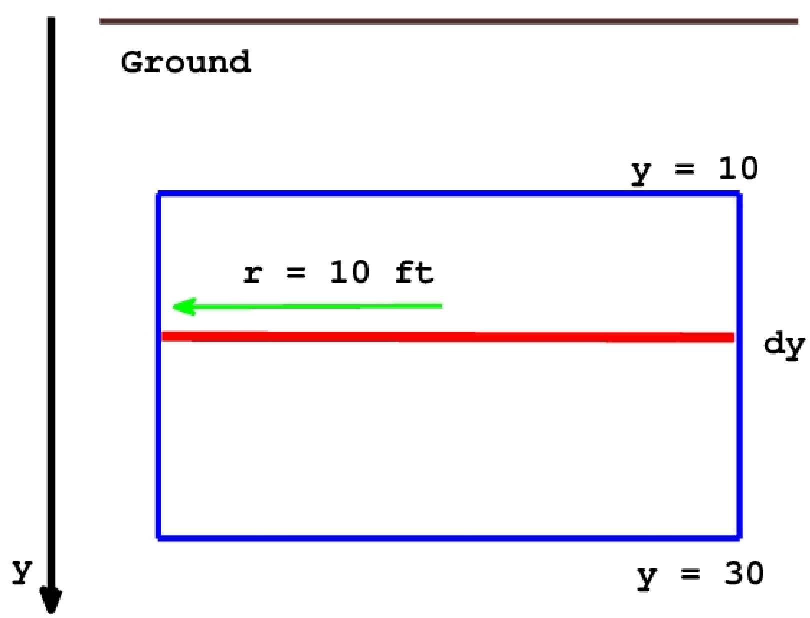

Figure 2: The cistern of interest for Problem 4. A differential slice of thickness dy is shown.