

Name: _____ Student ID: _____

Quiz 3

Directions: You will have 30 minutes to complete this quiz. Please show all of your work and mark your answers clearly. You may not use any extra resources during the quiz: not your notes, not your book, not a cell phone, not a calculator. Good luck.

1. (4 points) If a force of 90 N stretches a spring 1 m beyond its natural length, how much work is required to stretch the same spring 5 m beyond its natural length? HINT: Your answer should have units of Joules (denoted by a capital J).

Solution: First, we find the spring constant k for Hooke's law. We know that $F = kx$, where F is the force applied and x is the distance the spring is stretched. Substituting in the numbers given in the problem statement we see

$$90 \text{ N} = k \cdot 1 \text{ m.}$$

1 point for finding spring constant.

Solving this for k gives $k = 90 \frac{\text{N}}{\text{m}}$. Now that we know the spring constant, we can calculate the work done to stretch the spring.

Work is defined as $W = \int_a^b F(x) dx$. Therefore

$$W = \int_0^5 kx dx = \int_0^5 90x dx = 45x^2 \Big|_0^5 = 45 \cdot 25 = 1125 \text{ J (or N}\cdot\text{m).}$$

1 point for correct answer.

2 points for setting up work integral

2. (6 points) A thin plate covers the region bounded below by the parabola $y = x^2$ and above by the line $y = x$. The plate's density at a point (x, y) is given by $\delta(x) = 10x$. Find M_y , the moment of the plate about the y-axis (HINT: We defined $M_y = \int x dM$).

Solution: While reading this solution, please reference the figure on the last page.

We cut the plate into thin strips of thickness dx . Each of these strips has thickness $h(x) = x - x^2$. Therefore each has an area $dA = h(x)dx = (x - x^2)dx$. To find the mass of each strip, we multiply its area by the density function $\delta(x)$. Therefore $dM = 10x(x - x^2)dx$. To find the limits of integration, we set the top and bottom curves equal to each other, then solve for x .

$$x = x^2 \Rightarrow x = 1 \text{ and } x = 0.$$

3 points for correct dM

Finally, we multiply the moment arm to each strip (x), by the mass dM and integrate

$$\begin{aligned} M_y &= \int_0^1 x dM \\ &= \int_0^1 10 \cdot x \cdot x(x - x^2) dx \\ &= 10 \int_0^1 (x^3 - x^4) dx \\ &= 10 \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1 \\ &= 10 \left(\frac{1}{4} - \frac{1}{5} \right) \\ &= \frac{1}{2} \end{aligned}$$

2 points for correct Integral

1 point for calculating correct answer.

3. (4 points) Using the definition of a^x discussed in class, calculate the derivative of the function

$$f(x) = a^x \quad \text{where } a > 0.$$

HINT: You may only assume that $\frac{d}{dx}e^x = e^x$. Otherwise, show all of your work. Simply writing down the answer will get you no credit.

Solution: As discussed in class, the we defined exponentiation as

$$a^x = (e^{\ln(a)})^x = e^{x \ln(a)}. \quad \text{2 points for knowing the definition of } a^x$$

Using this definition, we calculate

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{x \ln(a)}]$$

$$= e^{x \ln(a)} \cdot \frac{d}{dx}[x \ln(a)]$$

$$= e^{x \ln(a)} \cdot \ln(a)$$

$$= \ln(a)a^x$$

} 2 points for using chain rule to differentiate properly.

4. (6 points) A cylindrical cistern is 20 ft tall and has a radius of 10 ft. The top of the cistern is buried 10 ft below ground level. If it is filled with a fluid of weight density w , how much work is required to pump all the fluid up to ground level.

Solution: While reading this solution, please reference the figure on the last page.

First, we define the y -axis to originate at ground level, and proceed downward. This means that the top of the cistern is at $y = 10$ and the bottom of the cistern is at $y = 30$. There are other ways to define the axis, but this way will lead to the simplest calculation. We cut the tank into horizontal slices (or cross-sections) perpendicular to the y -axis. Each cross-section has a thickness dy . Each cross-section has a volume $dV = A(y)dy$. To find a formula for $A(y)$, we note that each cross section is a circle of radius 10 ft. Therefore, $A(y) = \pi r^2 = 100\pi$. The weight of fluid in each cross section (and therefore the force required to lift it) is the fluid weight density times the volume: $w dV = 100\pi w dy$. Finally, we calculate the distance that each cross-section of fluid must be pumped. The cross-section must be pumped from its initial height y to the origin of our axis, for a total distance of $D = y$. So, we calculate the total work required as

$$W = \int_a^b w D dV$$

$$= \int_{10}^{30} 100\pi w y dy$$

$$= 50\pi w \int_{10}^{30} 2y dy$$

$$= 50\pi w [y^2]_{10}^{30}$$

$$= 50\pi w (900 - 100)$$

$$= 40000\pi w$$

} 3 points for setting up correct Integral

} 1 point for correct evaluation of Integral

This may change depending on how you set your axis

2 points for finding dV & distance travelled.

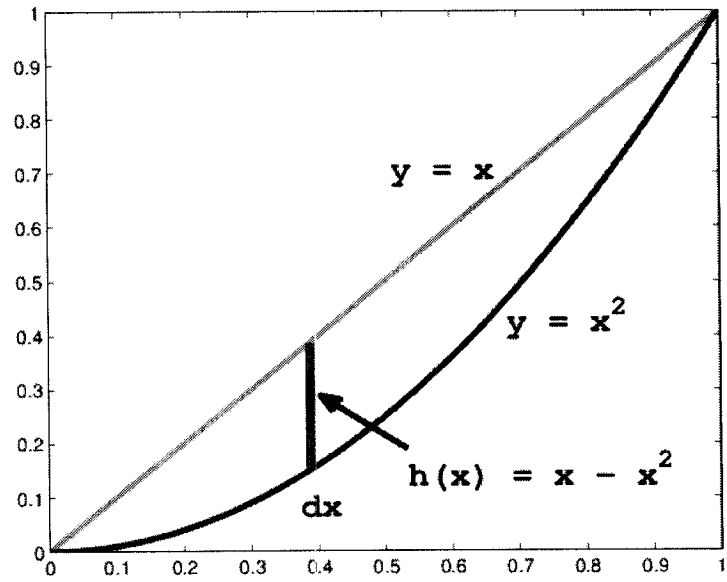


Figure 1: The region of interest for Problem 2. A differential slice of thickness dx is shown.

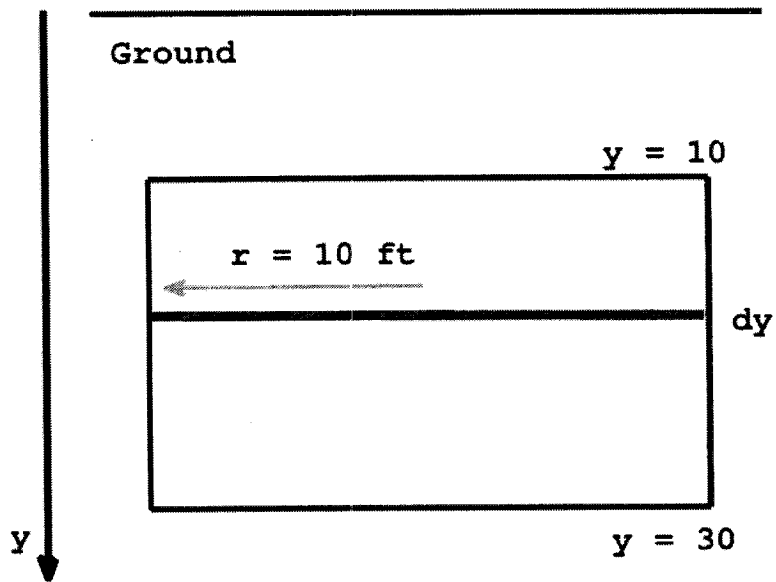


Figure 2: The cistern of interest for Problem 4. A differential slice of thickness dy is shown.

