

Homework 2

Due: Tuesday Sept. 29 (In Class)

A: (Suggested) Read Chapter 3 in *Mathematical Models in Biology*.

1. In 1973 Varley, Gradwell and Hassell proposed the following model of a population with a density dependent reproduction rate

$$p_{n+1} = \frac{\lambda}{\alpha} p_n^{1-b}, \quad (1)$$

which can be rewritten as

$$p_{n+1} = \left(\frac{1}{\alpha} p_n^{-b} \right) (\lambda p_n). \quad (2)$$

- (a) Interpret the two terms in equation 2 in your own words.
- (b) Find the equilibrium population. Under what conditions is it stable? Sketch the bifurcation diagram.
- (c) What happens to the population when the stability conditions are violated? Include a cobweb diagram of an illustrative case.
2. Consider the tumor growth model we discussed in class

$$C_{n+1} = (m\sigma(C_n) + p) C_n. \quad (3)$$

Remember that p represented the base-line survival rate of mature cancer cells, m represents the per-capita number of new cancer cells produced in the tumor, and σ is the *tumor size dependent* survival rate which represents the body's ability to fight a tumor. We decided that for this model to be reasonable $\sigma(0) = 1$, and that as $C \rightarrow \infty$, $\sigma(C) \rightarrow 0$. In class we discussed $\sigma(C) = 1/(1 + \alpha C)$. Another function that behaves this way is

$$\sigma(C) = e^{-\alpha C}. \quad (4)$$

Using this new model of immune response, what can we say?

- (a) When will there be a positive equilibrium tumor size? Interpret this condition in words.
- (b) When will the positive equilibrium tumor be a stable equilibrium? When will no tumor (tumor size zero) be a stable equilibrium?
- (c) Assume that $p = 1/2$. Explore values of m from 0 to 30 and include cobweb plots of each behavior you see. Sketch a bifurcation diagram as a function of m . Label all bifurcations.

3. Suppose that a population of parasitic wasps (denoted with P_n) and host insects (denoted H_n) follow the Nicholson-Bailey model, *except* that each year some of the hosts manage to find safe refuge and therefore cannot be attacked by wasps.
- (a) Suppose that a set fraction θ of the hosts manage to find safe refuge. Write the equations in this case. Find the equilibria and determine if they are stable.
 - (b) Now assume that the “refuge” can only hold a certain number (S) of hosts, and therefore $H_n - S$ hosts are subject to attack by wasps. Also assume that the hosts which find refuge have a per capita reproduction of 1, while those that do not find refuge have the standard reproduction value λ . Find the equilibria and stability. How is this behavior different than in part (a).
4. Suppose that we have a parasitoid/host system where the host organism has an age structure that we wish to model. Juveniles are not vulnerable to attack by parasitoids, but adults are. Suppose that the attacking rate of vulnerable hosts and the growth of parasitoids follows the Nicholson-Bailey model. The adults that escape attack produce (on average) λ offspring. The fraction of juveniles that survive to become adults is given by σ .
- (a) Write down the equations that describe this system (there are 3).
 - (b) Find the non-zero fixed point and the Jacobian of the system at this fixed point.
 - (c) Are there parameters for which the determinant is less than 1 in magnitude? If so, do you think the fixed point is stable for these parameter values?