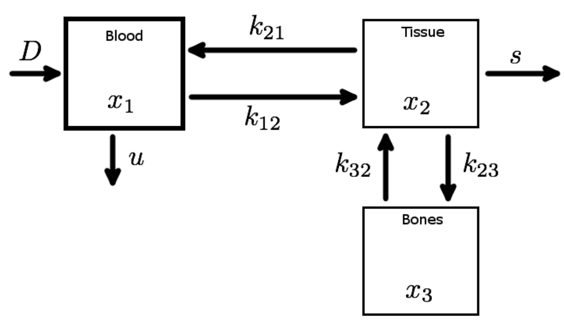
Math 5110

Homework 4
Due: Thursday Nov. 19 (By 5pm)

1. Read Chapter 6 in the textbook.
2. A semi-toxic chemical is ingested by an animal and enters its bloodstream at a constant rate D. It distributes within the body, passing from the blood to soft tissue to bones with (positive) rate constants indicated in the figure. It may be removed from the blood (and excreted as urine) and the soft tissue (through sweat) at rates u and s respectively. A model of this type was proposed by Batsschelet, *et. al* in 1979 to study the kinetics of lead in the human body. Let x_1, x_2, and x_3 represent the concentration of the chemical in the three resevoirs. From the diagram below, we may write the following equation for the evolution of chemical concentration in the blood,

$$\frac{dx_1}{dt} = D - ux_1 - k_{12}x_1 + k_{21}x_2. \tag{1}$$



   (a) Assuming that exchange between all three compartments is linear (as in the above equation), write equations for the evolution of x_2 and x_3.
   (b) Find the steady-state concentration of chemicals throughout the body.
   (c) Do you think that this steady state is stable? How would you find out?

3. In 1971 J.S. Griffith proposed the following model for the interaction of messenger RNA and protein. The concentration of messenger RNA is given by M and the concentration of the protein is given by by E. These concentrations evolve according to the equations

$$\frac{dM}{dt} = \frac{aKE^m}{1 + KE^m} - bM \tag{2}$$

$$\frac{dE}{dt} = cM - gE. \tag{3}$$

Here, m is an integer that reflects the order of the RNA producing reaction.

(a) Rescale these equations to arrive at the non-dimensional set of equations

$$\frac{d\tilde{M}}{d\tau} = \frac{\tilde{E}^m}{1 + \tilde{E}^m} - \alpha\tilde{M} \tag{4}$$

$$\frac{d\tilde{E}}{d\tau} = \tilde{M} - \beta\tilde{E}. \tag{5}$$

Give the non-dimensional parameters α and β in terms of the dimensional parameters a, K, b, c, and g. For the rest of this problem, analyze the non-dimensional equation.

(b) Find one steady state, and show that there is another steady state that is defined by an algebraic equation. Now set $m = 1$. What condition must the non-dimensional parameters satisfy for the second equilibrium to exist and be positive (physical). Draw a phase-plane diagram for both cases where this equilibrium does and does not exist. What happens to the concentrations $\tilde{M}$ and $\tilde{E}$?

(c) Now set $m = 2$. Find an algebraic equation that defines the steady state protein concentration. Show that for certain conditions on α and β, there are two positive physical equilibrium and if this condition is broken, there are none. Draw the phase diagram for each case. What happens to the concentrations $\tilde{M}$ and $\tilde{E}$?