Derivative Markets Module 1

Section 1: Introduction (to Derivative Markets)

Definition: A financial **derivative** is a financial contract between two people that has a value that is "derived" from the value of its *underlying asset.* The common theme with all derivatives is that, although we know the price of the underlying asset when the contract is opened (this is just the current price of the underlying asset), we do not know what its price will be when the contract is closed (this is called the **spot price** of the underlying asset **at expiration**). The person buying or purchasing the contract has a *long position in the derivative*, whereas the person selling the contract has a *short position in the derivative.* These are reverse positions of one another, in the sense that any profit made from one position is paid for by a loss from the other position. It's a zero-sum game.

Assumptions: (Most of this is very unrealistic.)
1. There is a market for everything. That is, we can buy and/or sell any number of contracts of any type of derivative at any time. We can loan or borrow any amount of money at any time and do so at the risk-free rate.
2. All positions are closed at expiration.
3. All obligations are honored. The risk that one party abandons the contract is called **credit risk**. We are assuming no credit risk.
4. Unless otherwise stated or implied, the underlying asset does not pay dividends.
5. Unless otherwise stated or implied, there are no fees, transaction costs, or other deposits required for trading derivatives.
6. Unless otherwise stated or implied, there are no arbitrage opportunities. This means that for each contract, the buyer will profit for some changes in the value of the underlying asset but will lose for other changes in the value of the underlying asset. Note that when the buyer profits, the seller loses, and vice-versa.

The **payoff** (at expiration) of a position is the amount of money exchanging hands at expiration. It does not depend on the initial cost (IC) of the position. The **profit** (at expiration) is the difference between the payoff at expiration and the accumulated value of the initial cost, denoted AVIC.

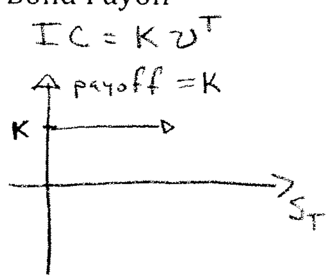
Profit = Payoff – AVIC

Unless otherwise stated, we determine AVIC using the risk-free interest rate. We capture payoff and profit information in graphs, called the payoff diagram and the profit diagram. The horizontal axis for these diagrams is the spot price at expiration. The graph of a short position is the reflection of the graph of the corresponding long position about the horizontal axis.

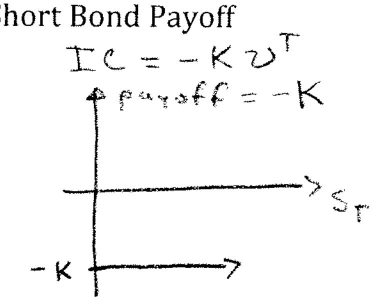
Bonds in the Context of Derivative Markets:
Unless otherwise stated, bond will mean a **zero-coupon bond**. For example, a Short Bond(K,T) position means we sell a zero-coupon bond with a redemption value of K that will be redeemed in T years. We sold the bond, and so our initial cost is $IC = -Kv^T$ (negative since we're receiving money), where v is the annual discount factor. Since we're paying the redemption value at expiration (remember we sold the bond), then we have payoff = $-K$ (constant with respect to the price of the underlying asset). Analogous statements hold for a Long Bond(K,T) position.

Payoff Diagrams for Bonds: (Since the payoff on a bond position is constant, the payoff diagram for a bond position is a horizontal line.)

Long Bond Payoff



Short Bond Payoff



Important Comments Regarding Bonds:

1. Since the initial cost (IC) equals the present value of the redemption value, then accumulated value of the initial cost (AVIC) sends us right back to the redemption value. Recalling that profit = payoff – AVIC, and using the bond payoff information above, we see that the profit from buying or selling a bond is 0.

2. Bond positions are equivalent to lending or borrowing money at the risk-free interest rate. For example, if we take a Long Bond(K,T) position, then we pay Kv^T today in return for receiving K at time T. This is equivalent to lending money. Likewise, a Short Bond(K,T) position is equivalent to borrowing.

3. Frequently we will be adding payoff diagrams. Since the payoff diagram for a long (short) bond position is a horizontal line at $y = K$ ($y = -K$), adding a bond to an existing position will result in shifting the payoff diagram for the existing position up (or down) by K units. Adding a bond to an existing position will not affect the profit of the position because of Comment 1.

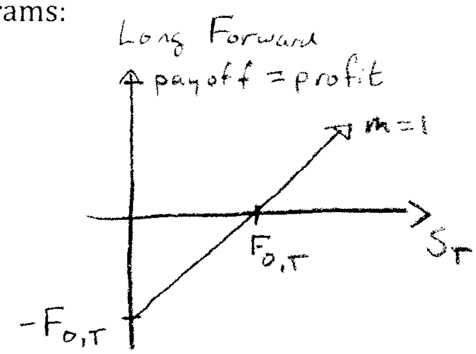
Section 2: Forwards

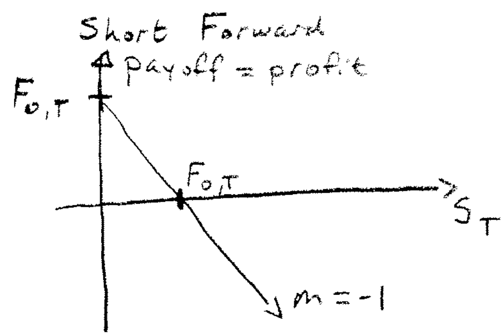
A **forward contract** is an agreement made today (at time 0) to buy or sell the underlying asset at a specified date in the future for a specified price. The initial cost of a forward contract is 0; that is, there is no money exchanged between the buyer and the seller at time 0. The specified future date is called the **expiration date** (at time T) and the specified price is called the **forward price**. The person with the long forward position has an obligation to *buy* the underlying asset on the expiration date for the forward price, regardless what the spot price at expiration is. The person with the short position is always obligated to be on the other side of the transaction, and so the person with the short forward position has an obligation to *sell* the underlying asset on the expiration date for the forward price.

Standard Notation:

- S_T is the spot price of the asset at time T (spot price at expiration)
- S_0 is the current price of the asset
- $F_{0,T}$ is the forward price

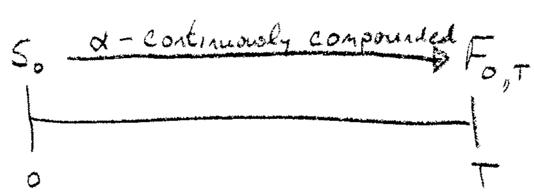
Long Forward Payoff = $S_T - F_{0,T}$ (= Long Forward Profit since IC = 0)
Short Forward Payoff = $F_{0,T} - S_T$ (= Short Forward Profit since IC = 0)
Diagrams:





Forward Premium (FP): $FP = \frac{F_{0,T}}{S_0}$

Annualized Forward Premium (sometimes referred to as cost of carry): This is the continuously compounded rate, α, which makes the current price equivalent to the forward price. Therefore, $S_0 e^{\alpha T} = F_{0,T} \quad \Rightarrow \quad \alpha = \frac{1}{T}\ln(FP)$



A **pre-paid forward contract** is a forward contract in which the underlying asset is delivered at the expiration date but the contract is paid for at time 0. Note that a no arbitrage assumptions means that the price paid at time 0 for a pre-paid forward contract is equivalent, using the risk-free interest rate, to the corresponding forward price at expiration for a forward contract on the same underlying asset.

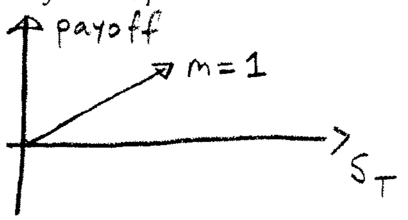
Standard Notation:

$F_{0,T}^{P}$ is the pre-paid forward price

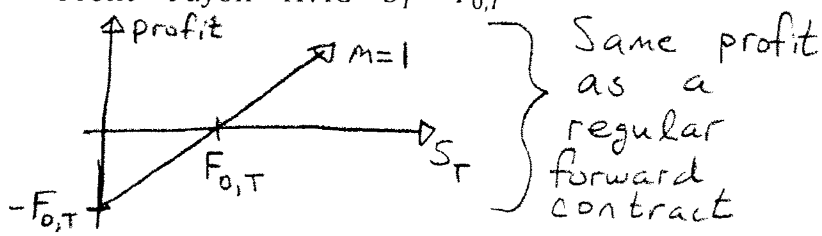
FACT: Under a no arbitrage assumption, $F_{0,T}^{P}$ is equivalent to $F_{0,T}$ using the risk-free interest rate. For example, if the risk-free interest rate is given as a continuously compounded rate, r, then $F_{0,T} = F_{0,T}^{P}\, e^{rT}$.

Long Pre-paid Forward Payoff and Profit: IC = $F_{0,T}^{P}$ and AVIC = $F_{0,T}$
(Obtain short diagrams by reflecting)

Payoff = S_T



Profit = Payoff – AVIC = $S_T - F_{0,T}$



Pre-paid forward prices and annualized forward premiums: In the following cases, assume r is the risk-free rate, continuously compounded.

Case 1: The underlying asset is a stock that does not pay dividends.

Then $F_{0,T}^{P} = S_0$ $(\alpha = r)$

Case 2: The underlying asset is a stock that pays discrete dividends

$F_{0,T}^{P} = S_0 - PV(\text{dividends})$ (no closed rule formula for α)

Case 3: The underlying asset is a stock that pays continuous dividends at dividend rate, δ. (The annualized forward premium, α, may be referred to as cost of carry.)

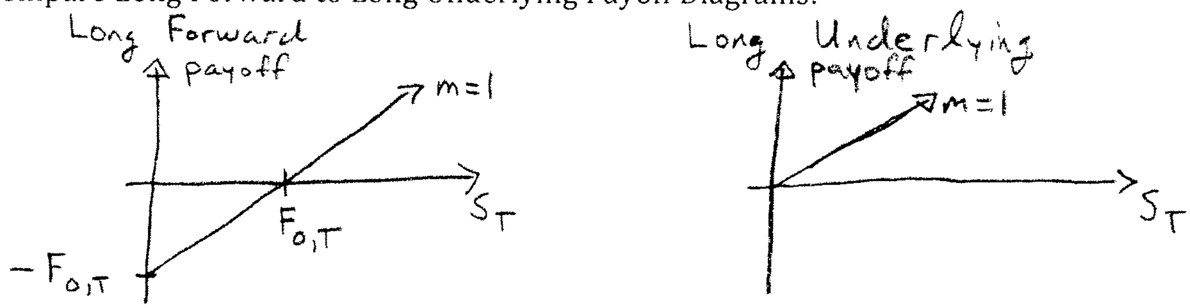
$F_{0,T}^{P} = S_0\, e^{-\delta T}$ $(\alpha = r - \delta)$

Do Numbers 1 through 10 of the Derivative Markets Module 1 Problems.

Section 3: Synthetic Forwards and Cash-and-Carry

Replicating a contract by using combinations of other positions is called creating a **synthetic** contract.

Compare Long Forward to Long Underlying Payoff Diagrams:



Notice that the long forward payoff is the long underlying payoff shifted down by the forward price units. Recall that we can shift a payoff diagram down by adding a short bond position. So we can get the long forward payoff diagram by taking a long underlying position plus a short bond position with redemption value K = $F_{0,T}$.

$$\text{Long Forward}(F_{0,T}) = \text{Long Underlying} + \text{Short Bond}(F_{0,T}, T)$$

Now recall that a short bond position is equivalent to borrowing money. The above equation implies that we can create a synthetic long forward by borrowing the money to buy the asset.

Arbitrage:
In the previous section we learned how to compute a no-arbitrage forward price. If a forward is available in the market such that the forward price is *not* our computed no-arbitrage forward price, then there is an arbitrage opportunity.

If the forward available in the market has a forward price greater than the no-arbitrage forward price, then the forward is expensive relative to the asset. Being expensive, *sell it.* The following cash-and-carry position will create arbitrage.

Cash-and-Carry: Long Underlying + Short Forward

If the forward available in the market has a forward price less than the no-arbitrage forward price, then the forward is cheap relative to the asset. Being cheap, *buy it.* Reverse the above position. The following reverse cash-and-carry position will create arbitrage.

Reverse Cash-and-Carry: Short Underlying + Long Forward

Do Numbers 11 and 12 of the Derivative Markets Module 1 Problems.

Section 4: Call Options

A (European) **call option** is an agreement in which the buyer of the call option has the *right*, but not the *obligation*, to buy the underlying asset at a specified date in the future (the expiration date), for a specified price. The specified price is called the **strike price**, or **exercise price**. If the spot price of the underlying asset at expiration is greater than the strike price, then the owner of the call will exercise the option; the owner of the call profits by buying the underlying asset for the strike price and then selling the asset for the greater market price. Otherwise, if the spot price of the underlying asset at expiration is less than the strike price, then the owner of the call will let the option expire without exercising the option. There is a monetary value associated with the call owner being able to let the option expire without exercising it. This value is called the **call premium**. The call premium depends on the strike price, K, and the time to expiration, T. It is denoted by Call(K,T) and it is the amount that the buyer of the call option pays the seller of the call option at time 0.

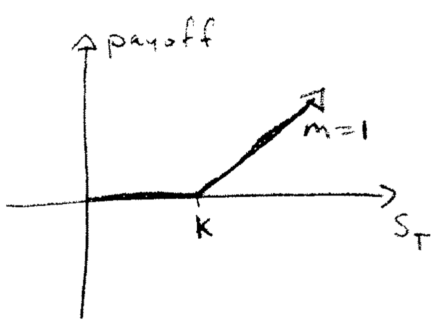
Remarks:

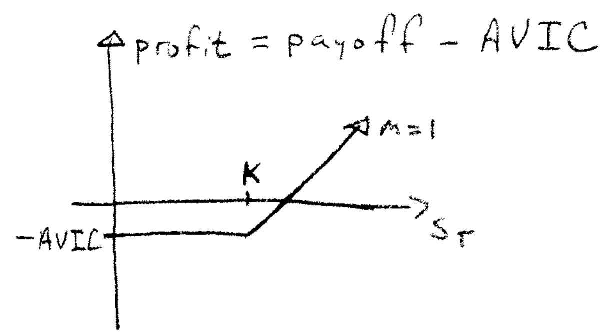
1. An option seller is said to be the option **writer** or to have **written the option**.

2. As indicated, we described a European option. An **American** option is defined similarly except that the owner can exercise the option at any time up to and including the expiration date. A **Bermudan** option is between a European and American option in the sense that this type of option can be exercised at several specified discrete dates (e.g. the end of each month) between purchase and expiration dates. Unless otherwise stated, all options are European.

Payoff and Profit Diagrams for Long Call Option Positions

Long Call(K,T) $IC = Call(K,T)$





$$\text{payoff} = \max(0, S_T - K)$$

Section 5: Put Options

A **put option** is an agreement in which the buyer of the put option has the right, but not the obligation, to *sell* the underlying asset at the expiration date for the strike, or exercise, price. If the spot price of the underlying asset at expiration is less than the strike price, then the owner of the put will exercise the option; the owner of the put profits by selling the underlying asset for the strike price and then buying the asset for the lower market price. Otherwise, if the spot price of the underlying asset at expiration is greater than the strike price, then the owner of the put will let the option expire without exercising the option. There is a monetary value associated with the put owner being able to let the option expire without exercising it. This value is called the **put premium**. Like the call premium, the put premium depends on the strike price, K, and the time to expiration, T. It is denoted by Put(K,T) and it is the amount that the buyer of the put option pays the seller at time 0.

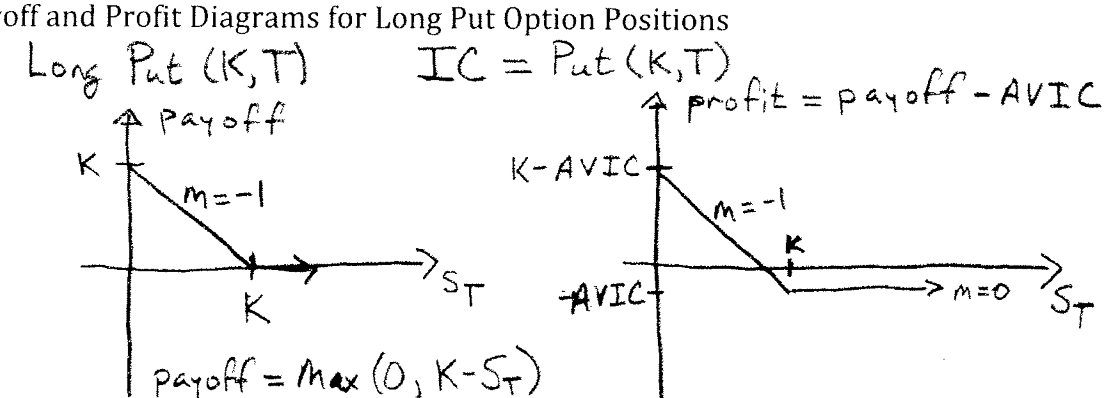
Remarks:

1. If the spot price is such that the owner would let the option expire without exercising it, then we say the option is **out of the money**. If the spot price at expiration is such that the option will be exercised then we say the option is **in the money**. If the spot price at expiration is equal to the strike price, then we say the option is **at the money**.

2. There are two values associated to option premiums, whether call premiums or put premiums; the **intrinsic value** and the **time value**. The intrinsic value is the amount by which the option in currently in the money. A time value is added to the intrinsic value to get the option premium. The longer until the option matures, the larger the time value of the option. Options that are out of the money have no intrinsic value, and option that are about to mature have no time value.

3. A long put position has the option to sell the underlying asset, and so it corresponds to a short underlying asset position. Inversely, a short put position corresponds to a long position with respect to the underlying asset.

Payoff and Profit Diagrams for Long Put Option Positions

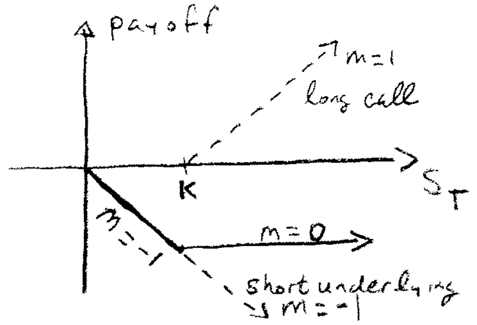


Do Numbers 13 through 18 of the Derivative Markets Module 1 Problems.

Section 6: Caps, Floors, and Covered Writing

Caps, covered calls, floors, and covered puts are basic compound positions that combine a position in the underlying asset with a position in a call or put option.

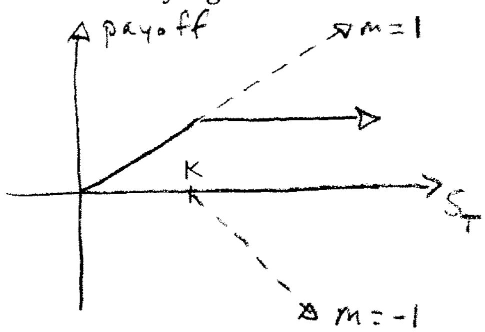
1. Cap(K,T) = Short Underlying + Long Call(K,T)



$$IC = -S_0 + Call(K,T)$$

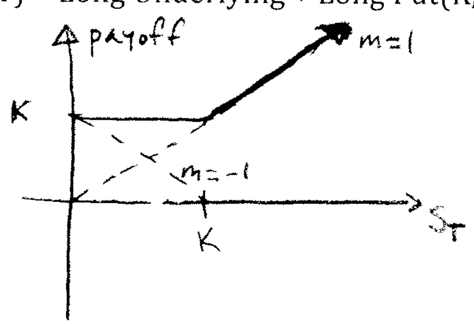
Solid line represents payoff from combined position.

2. Covered Call(K,T) = Reverse Cap(K,T) = Long Underlying + Short Call(K,T)
Generally, **covered writing** is selling an option while having a position in the underlying asset. **Naked writing** refers to selling an option while not having a position in the underlying asset.



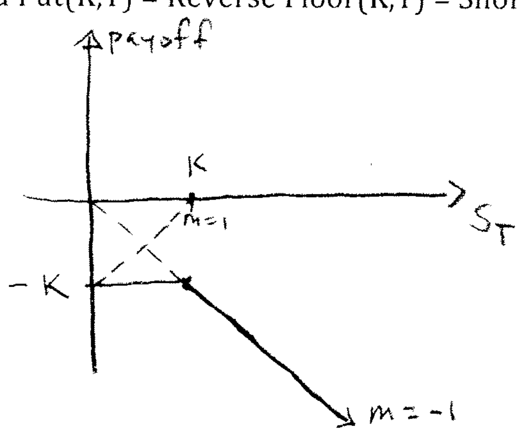
$$IC = S_0 - Call(K,T)$$

3. Floor(K,T) = Long Underlying + Long Put(K,T)



$$IC = S_0 + Put(K,T)$$

4. Covered Put(K,T) = Reverse Floor(K,T) = Short Underlying + Short Put(K,T)

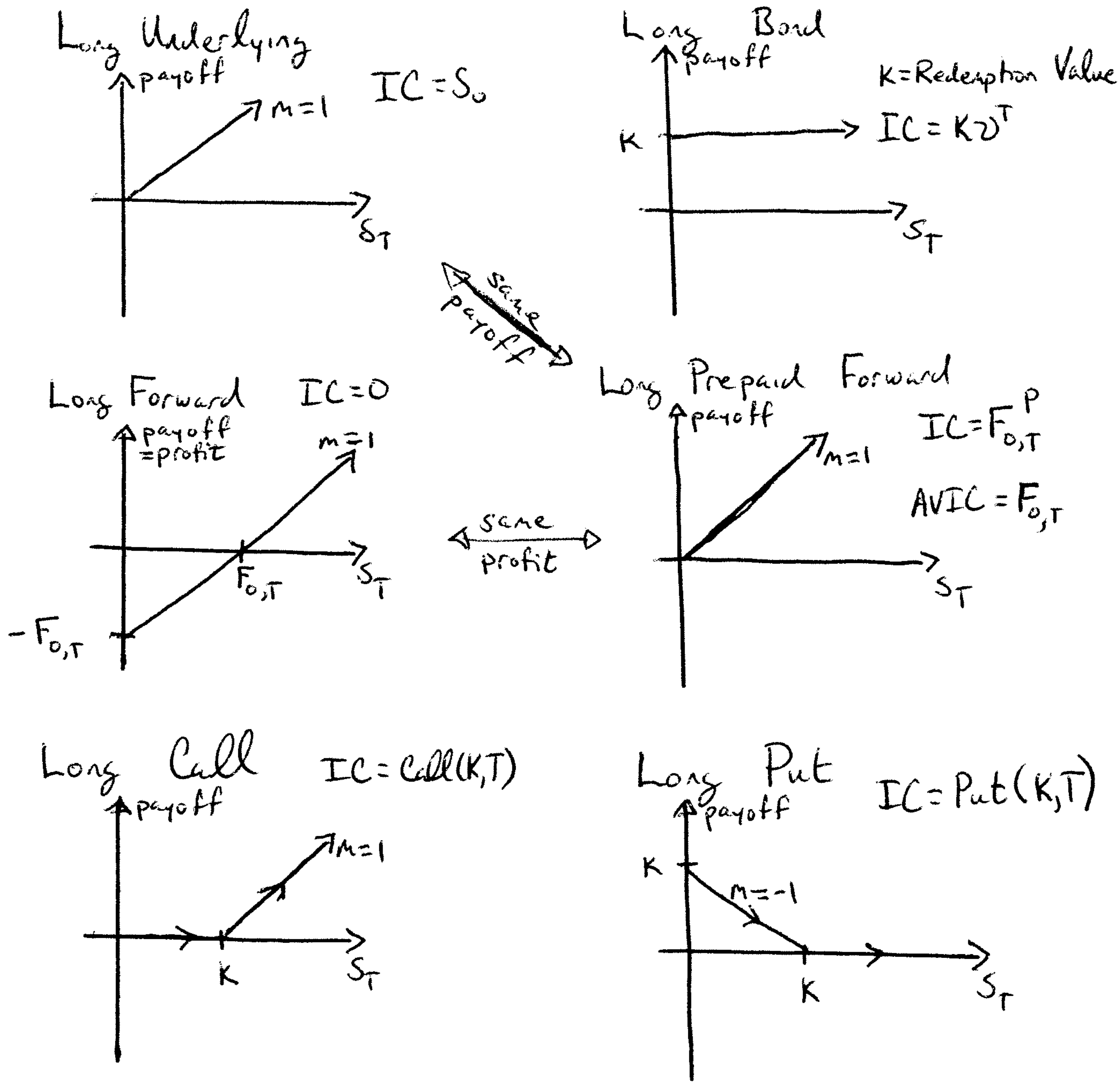


$$IC = -S_0 - Put(K,T)$$

Section 7: Summary

The building blocks of derivative markets are stocks, bonds, forwards (including prepaid forwards), call options, and put options. Below are the graphs of the payoff of the long position of these derivatives. The graphs of the payoff of the short position of these derivatives are the reflection about the horizontal axis, which is representing the unknown spot price at expiration of the underlying asset. Profit is always equal to payoff minus the accumulated value of the initial cost at the risk-free interest rate.

profit = payoff − AVIC



Important topics discussed in Module 1 include:

Forward premium: This equals the ratio of the forward price to the current price of the asset.

Annualized Forward Premium (sometimes called **Cost-of-Carry**): This is the continuously compounded interest rate that makes the (time 0) current stock price equivalent to the (time *T*) forward price.

Prepaid Forwards: A forward contract is an agreement made at time 0 in which the purchaser of the contract will pay the agreed upon forward price for the underlying asset, and take delivery of the underlying asset, at some future time. A prepaid forward is a slight variation of a forward contract in which payment is made at time 0 but delivery is not taken until time *T*. The prepaid forward price is equal to:

(1) $F_{0,T}^{P} = S_0$ if the underlying asset pays no dividends
(2) $F_{0,T}^{P} = S_0 - PV$(dividends between times 0 and T) if the underlying asset pays discrete dividends
(3) $F_{0,T}^{P} = S_0 e^{-\delta T}$ if the underlying asset pays dividends at the continuous rate δ.

Under a no-arbitrage assumption, the time 0 prepaid forward price is equivalent to the time *T* forward price, using the risk-free interest rate. For example, if the risk-free interest rate is given as a continuously compounded interest rate, *r*, then $F_{0,T} = F_{0,T}^{P} e^{rT}$. Notice that in this case, if the underlying asset pays dividends continuously at rate δ, then the annualized forward premium (cost-of-carry) becomes $\alpha = r - \delta$.

Replicating a contract using a combination of other positions creates a **synthetic** contract. For example:

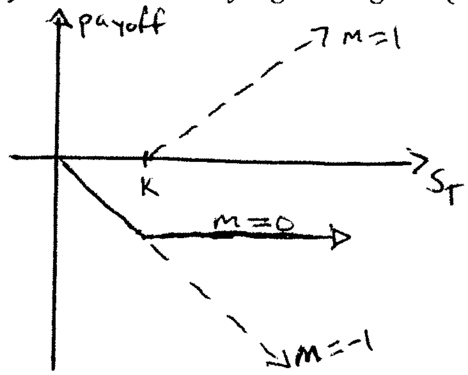
(1) (**Cash-and-Carry**) Long Underlying + Short Forward is equivalent to a long bond position
(2) (**Reverse Cash-and-Carry**) Short Underlying + Long Forward is equivalent to a short bond position

These are synthetic contracts that can be entered into in order to attain arbitrage when the forward is mispriced.

A (European) **call option** is an option to buy the underlying asset at some future time *T* for some agreed upon price *K*. A (European) **put option** is an option to sell the underlying asset at some future time *T* for some agreed upon price *K*.

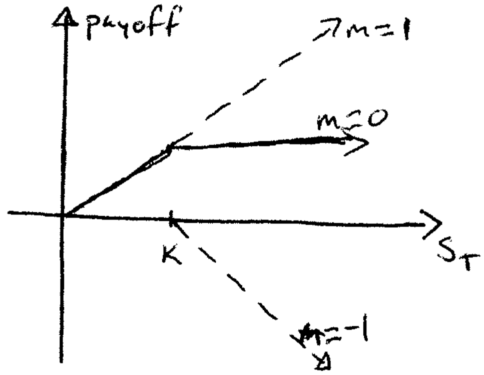
Basic Compound Positions:

Cap(*K*,*T*) = Short Underlying + Long Call(*K*,*T*)



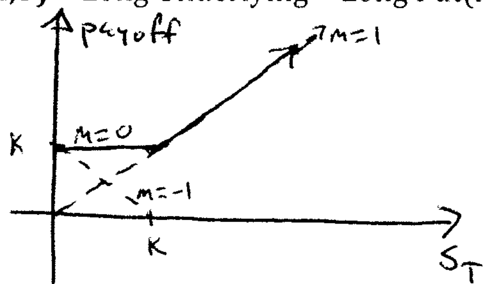
$$IC = -S_0 + Call(K,T)$$

Covered Call = Long Underlying + Short Call(*K*,*T*) (Reverse Cap)



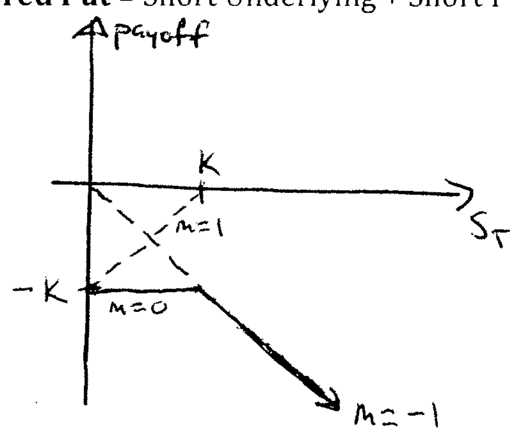
$$IC = S_0 - Call(K,T)$$

Floor(*K*,*T*) = Long Underlying + Long Put(*K*,*T*)



$$IC = S_0 + Put(K,T)$$

Covered Put = Short Underlying + Short Put(*K*,*T*) (Reverse Floor)



$$IC = -S_0 - Put(K,T)$$

Derivative Markets Module 1 Problems

1. Andy takes a long position on a 2-year forward contract with a forward price of 50. The spot price at expiration of the underlying asset is 42. Determine Andy's profit (at expiration).

2. Cathy takes a short position on a 6-month forward contract with a forward price of 75. Cathy's payoff at expiration is 7. Determine the spot price at expiration of the underlying asset.

3. Chuck buys a pre-paid forward contract that expires in 2 years. The risk-free interest rate is 4% annual effective. At expiration, the spot price of the underlying asset is 118.16 and Chuck realizes a profit of 10. Determine Chuck's initial cost.

4. Mary takes a short position on a 1-year forward contract. If the spot price at expiration of the underlying asset is S then Mary has a profit of 10, but if the spot price at expiration is $1.25S$ then Mary has a profit of -10. Determine the forward price.

5. Ellen sells 3 forward contracts on an underlying asset. If the spot price at expiration is 100, then Ellen has a profit of -15. Determine the forward price.

6. A stock currently sells for 50. A 2-year forward contract has forward price equal to 60. Determine the forward premium and the annualized forward premium.

7. A stock currently sells for 85. The risk-free interest rate is 2% annual effective. Under a no arbitrage assumption, determine the forward price for a 3-year forward contract, and determine the annualized forward premium for the contract.

8. A stock currently sells for 100. The stock pays semiannual dividends of 1.25 with the next dividend payable in 6 months. The risk-free interest rate is 3% compounded quarterly. Under a no arbitrage assumption, determine the annualized forward premium for a 2-year forward contract.

9. A stock currently sells for 90. The stock pays continuous dividends at the constant rate of $\delta = 0.02$. The risk-free interest rate is 4% compounded quarterly. Under a no arbitrage assumption, determine the annualized forward premium for a 3-year forward contract.

10. A stock currently sells for 50. The stock pays continuous dividends at the constant rate of $\delta = 0.02$. The risk-free interest rate is 4% compounded continuously. Under a no arbitrage assumption, determine the cost of carry for a 3-year forward contract.

11. A stock currently sells for 50. The risk-free interest rate is 3% compounded continuously.

a. A 2-year forward contract has forward price equal to 60. Describe a position that would guarantee a profit, and determine the amount of the profit.

b. A 2-year forward contract has forward price equal to 52. Describe a position that would guarantee a profit, and determine the amount of the profit.

12. A 1000 face value zero-coupon government bond, redeemable in 3 years, sells for 925. A 3-year forward contract on a stock index has forward price equal to 700. A reverse cash-and-carry position guarantees a profit of 2.70. Determine the current price of the index.

13. A 1-year 60-strike call option on a stock is selling for 5.80. The risk-free interest rate is 2% annual effective. Chris takes a long position on this call option. Determine Chris's profit if the market price of the stock in one year is 63.

14. The risk-free interest rate is 3% compounded continuously.

a. A 2-year 50-strike put is purchased for 4.40. Determine the payoff and profit if the stock is selling for 54 after two years.

b. Redo part a. if the stock is selling for 48 after two years.

c. A 2-year 50-strike call is purchased for 7.31. Determine the payoff and profit if the stock is selling for 54 after two years.

d. Redo part c. if the stock is selling for 48 after two years.

15. A 2-year 80-strike call option on a stock is selling for 7.50. The risk-free interest rate is 2% compounded continuously. Mike sells 1 such call option.

a. Determine Mike's payoff and profit if the spot price at expiration is 95.
b. Determine Mike's payoff and profit if the spot price at expiration is 85.
c. Determine Mike's payoff and profit if the spot price at expiration is 75.

16. A 2-year 60-strike put option on a stock is selling for 6.20. The risk-free interest rate is 1% compounded continuously. John takes a short position on this put option.

a. Determine John's payoff and profit if the spot price at expiration is 65.
b. Determine John's payoff and profit if the spot price at expiration is 55.
c. Determine John's payoff and profit if the spot price at expiration is 45.

17. A 2-year 80-strike call option on a stock is selling for 7. The risk-free interest rate is 2% compounded continuously. David sells 4 such call options.

a. Determine David's payoff and profit if the spot price at expiration is 95.
b. Determine David's payoff and profit if the spot price at expiration is 85.
c. Determine David's payoff and profit if the spot price at expiration is 75.

18. A 1-year 60-strike put option on a stock is selling for 6. The risk-free interest rate is 2% annual effective. Steve sells X such put options. If the strike price at expiration is 55, then Steve's profit is 3.36. Determine X.

19. A stock is currently selling for 55. A 1-year 60-strike call is selling for 5 and a 1-year 60-strike put is selling for 8.25. The annual effective risk-free interest rate is 3%. Determine the payoff and profit for each of the following positions at the given spot price at expiration.

a. Cap(60,1); $S_1 = 63$

b. Cap(60,1); $S_1 = 58$

c. Covered Call(60,1); $S_1 = 63$

d. Covered Call(60,1); $S_1 = 58$

e. Floor(60,1); $S_1 = 67$

f. Floor(60,1); $S_1 = 54$

g. Covered Put(60,1); $S_1 = 72$

h. Covered Put(60,1); $S_1 = 48$

i. 5*Cap(60,1); $S_1 = 50$

j. 3*Floor(60,1); $S_1 = 70$

20. A stock is currently selling for 80. A 2-year 80-strike call is selling for 10.58 and a 2-year 80-strike put is selling for 9. The continuously compounded risk-free interest rate is 1%. An investor enters into 4 covered call positions. Determine the break-even point for the investor.