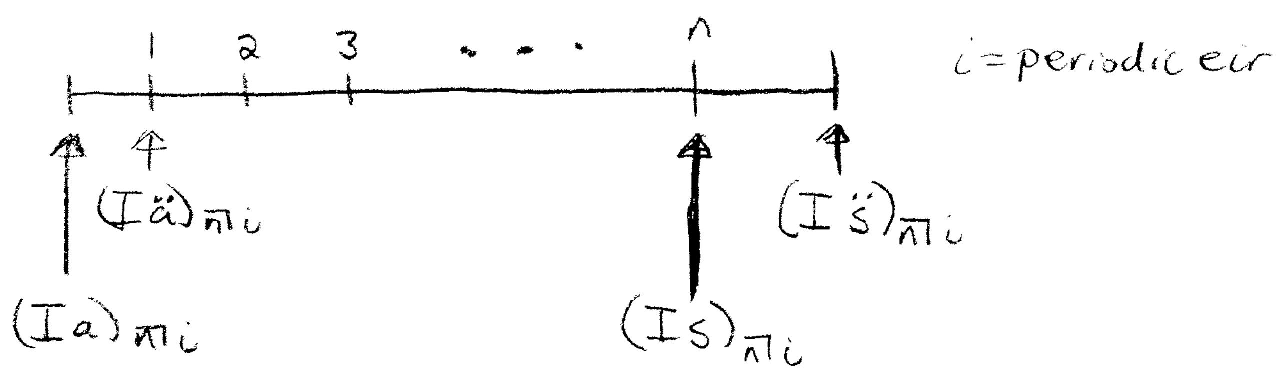
Section 6: Arithmetic Annuities

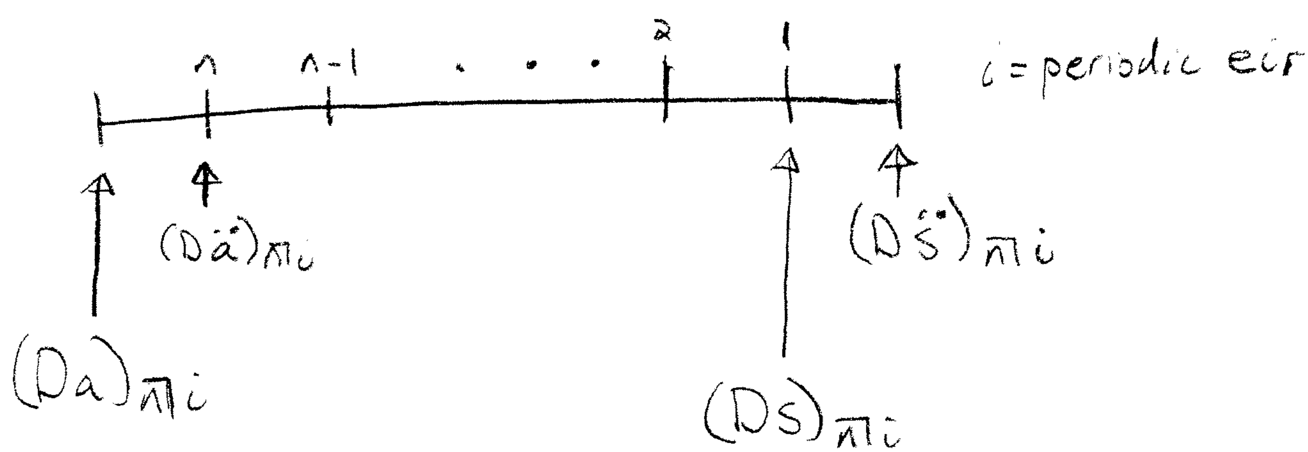
A sequence of terms forms an arithmetic progression if there is a "common difference" between consecutive terms of the sequence. This means that given any term in the sequence, we can get the next term by adding the common difference, denoted by d. Note that d may be negative, in which case we get from one term to the next by subtracting the common value.

An **arithmetic annuity** is an annuity for which the payments form an arithmetic progression. Unlike geometric annuities, there are special actuarial annuity symbols for the present and accumulated values of arithmetic annuities, which we'll get to below. Note that if $d < 0$ then the payments are decreasing, whereas if $d > 0$ the payments are increasing. If the first payment is 1 and $d = 1$, the payments are 1, 2, 3, ... n, and we call this annuity a basic increasing annuity. If the first payment is n and $d = -1$, the payments are n, $n - 1$, ..., 1, and we call this annuity a basic decreasing annuity.

Timelines and Notation:
(Basic Increasing Annuity)



(Basic Decreasing Annuity)



VEP's and CRF's for Basic Increasing and Decreasing Annuities

$$\boxed{(Ia)_{\overline{n|}} \overset{VEP}{=} v + 2v^2 + \cdots + nv^n \overset{CRF}{=} \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}}$$

$$(I\ddot{a})_{\overline{n|}} \overset{VEP}{=} 1 + 2v + 3v^2 + \cdots + nv^{n-1} \overset{CRF}{=} \frac{\ddot{a}_{\overline{n|}} - nv^n}{d} = (Ia)_{\overline{n|}} \cdot (1+i)$$

$$(Is)_{\overline{n|}} \overset{VEP}{=} (1+i)^{n-1} + 2\cdot(1+i)^{n-2} + \cdots + n \overset{CRF}{=} \frac{\ddot{s}_{\overline{n|}} - n}{i} = (Ia)_{\overline{n|}} \cdot (1+i)^n$$

$$(I\ddot{s})_{\overline{n|}} \overset{VEP}{=} (1+i)^n + 2\cdot(1+i)^{n-1} + \cdots + n\cdot(1+i) \overset{CRF}{=} \frac{\ddot{s}_{\overline{n|}} - n}{d} = (Is)_{\overline{n|}} \cdot (1+i)$$

$$\boxed{(Da)_{\overline{n|}} \overset{VEP}{=} nv + (n-1)\cdot v^2 + \cdots + v^n \overset{CRF}{=} \frac{n - a_{\overline{n|}}}{i}}$$

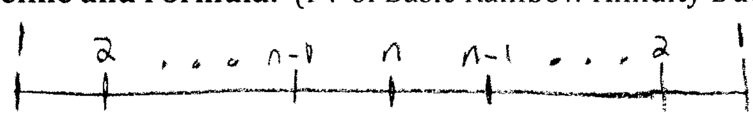
$$(D\ddot{a})_{\overline{n|}} \overset{VEP}{=} n + (n-1)\cdot v + \cdots + v^{n-1} \overset{CRF}{=} \frac{n - a_{\overline{n|}}}{d} = (Da)_{\overline{n|}} \cdot (1+i)$$

$$(Ds)_{\overline{n|}} \overset{VEP}{=} n\cdot(1+i)^{n-1} + (n-1)\cdot(1+i)^{n-2} + \cdots + 1 \overset{CRF}{=} \frac{n(1+i)^n - s_{\overline{n|}}}{i} = (Da)_{\overline{n|}} \cdot (1+i)^n$$

$$(D\ddot{s})_{\overline{n|}} \overset{VEP}{=} n\cdot(1+i)^n + (n-1)\cdot(1+i)^{n-1} + \cdots + (1+i) \overset{CRF}{=} \frac{n(1+i)^n - s_{\overline{n|}}}{d} = (Ds)_{\overline{n|}} \cdot (1+i)$$

Note that by knowing the two boxed formulas, we can easily derive the others by using the relationship in the last equality of each formula. The following two formulas are used often on exams.

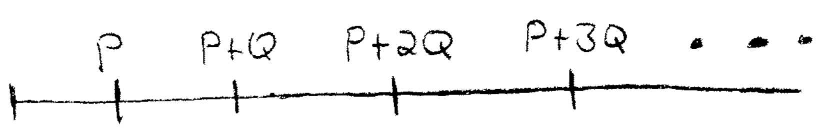
Timeline and Formula: (PV of Basic Rainbow Annuity Due)



peak = n
i = periodic eir

$$PV = (\ddot{a}_{\overline{n|}i})^2$$

Timeline and Formula: (PV of General Increasing Perpetuity Immediate)



i = periodic eir

$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

Module 2 Section 6 Problems:

1. Determine $12v + 11v^2 + \cdots + v^{12}$ using $i = 0.03$.

2. Determine $v + 2v^2 + 3v^3 \cdots + 12v^{12}$ using $i = 0.03$.

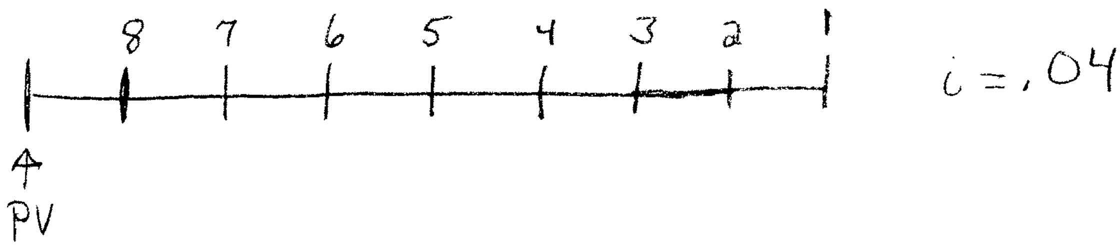
3. Determine $(1.025)^{25} + 2 \cdot (1.025)^{24} + 3 \cdot (1.025)^{23} \cdots + 25 \cdot (1.025)$

4. Determine $6 + 7v + 8v^2 + \cdots + 27v^{21}$ using $i = 0.05$

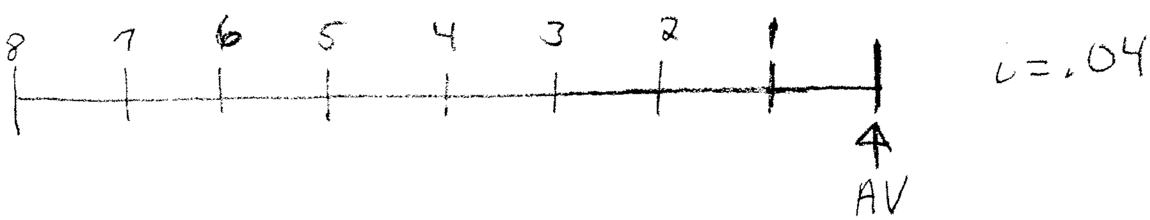
5. Determine $37 + 42(1.07) + 47(1.07)^2 + \cdots + 92(1.07)^{11}$

For Numbers 6 through 14, determine the value of the given annuity at the given valuation date using the given interest rate.

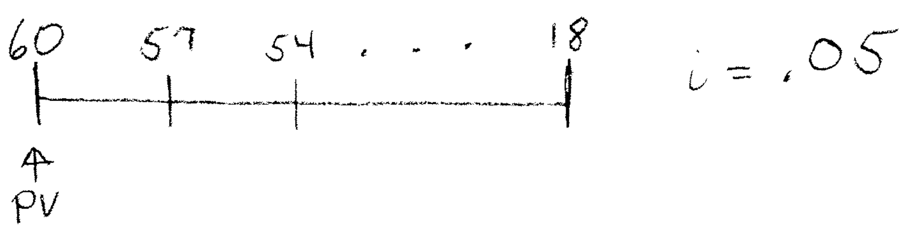
6.



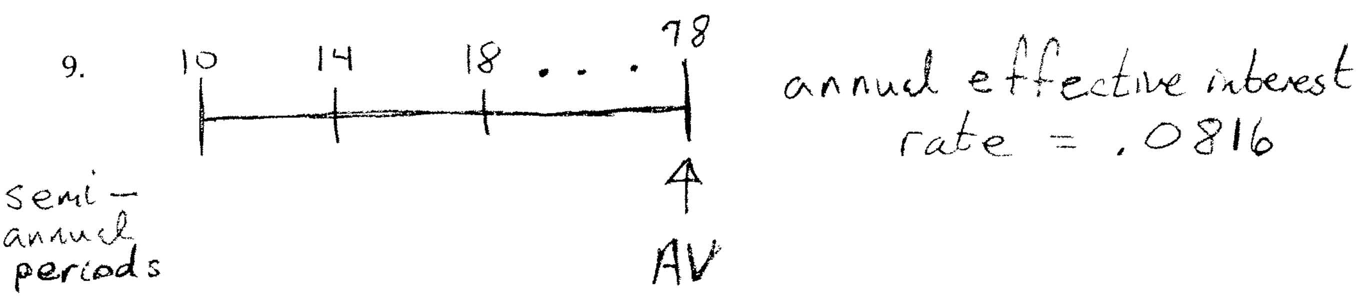
7.



8.

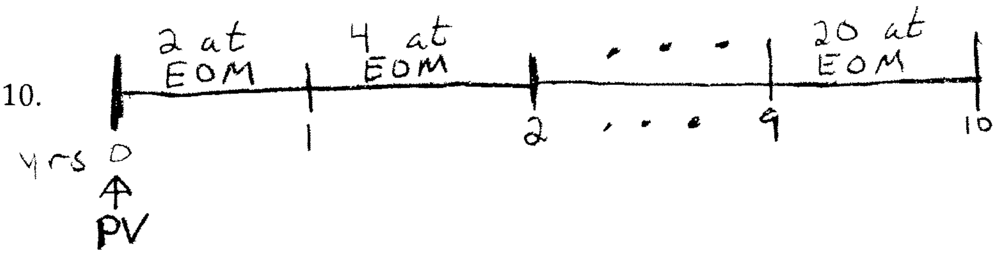


9.



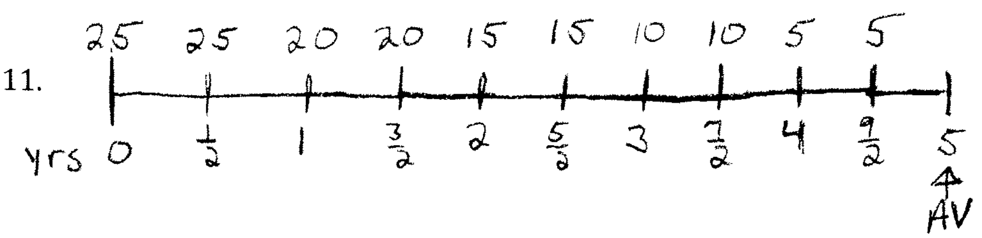
EOM - End of Month

10.



$i = .05$

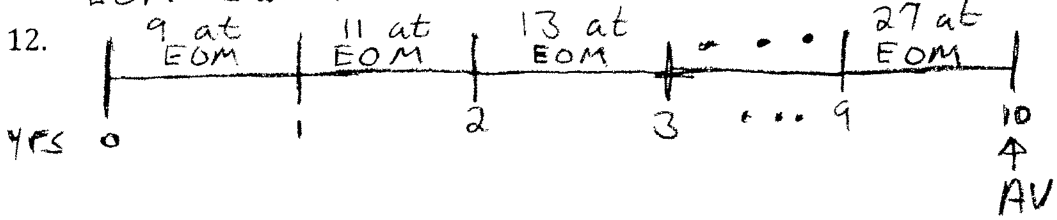
11.



annual effective interest rate = .06

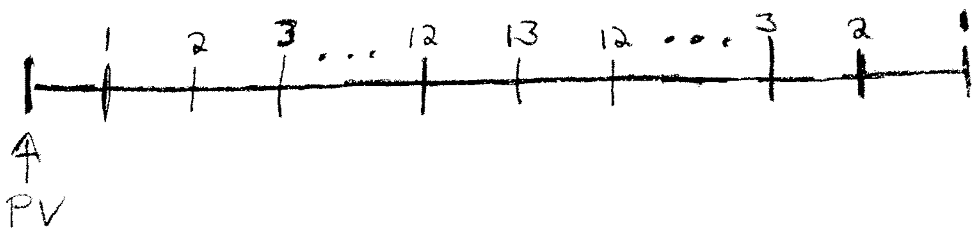
EOM - End of Month

12.



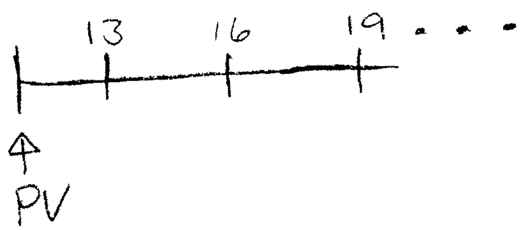
annual effective interest rate = .05

13.



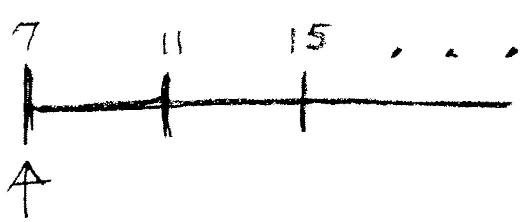
periodic eir = .03

14.



periodic eir = .025

15. Determine the periodic effective interest rate, *i*, given:



PV = 10557 using i

Answers to Module 2 Section 6 Problems

1) 68.1999
2) 61.2022
3) 410.4800
4) 201.4153
5) 1225.1339
6) 31.6814
7) 45.0925
8) 454.5612
9) 1020.9954
10) 966.4356
11) 183.5394
12) ~~1594.2218~~ 2654.7639
13) 116.4953
14) 5320
15) 0.02