Module 3

Section 1: Loan Repayment – Amortization Method

Terminology and Notation:

L – Loan Amount
i – Periodic Effective Loan Interest Rate
n – Number of Periodic Payments
R_k – Amount of the k^{th} Payment

I_k – Amount of the k^{th} payment that pays interest on the loan
P_k – Amount of the k^{th} payment that repays principal

NOTE: $R_k = P_k + I_k$ $\quad L = \sum_{k=1}^{n} P_k$

B_k – Balance Immediately After the k^{th} Payment

Determining Loan Balances
We can determine loan balances in two ways:

$B_k \overset{Ret}{=} AV(L) - AV(\text{Past Payments})$ $\quad$ *Ret* – Retrospective

$B_k \overset{Pro}{=} PV(\text{Remaining Payments})$ $\quad$ *Pro* – Prospective

Determining the amount of interest and/or the amount of principal in each payment:
Use the following:

$$I_k = i \cdot B_{k-1} \qquad P_k = R_k - I_k$$

Important Remarks:

1. Amount of principal repaid during a period, say from time k to time m ($k < m$)
   = balance at end of period – balance at beginning of period
   = $\sum_{i=k+1}^{m} P_i = B_k - B_m$

2. Amount of interest paid during a period, say from time k to time m ($k < m$)
   = total amount paid during period – amount of principal repaid during period
   = $\sum_{i=k+1}^{m} R_i - (B_k - B_m)$

Basic Relationships for Level Payment ($R_k = R$ for all k) **Amortizations**:

$L = B_0 = Ra_{\overline{n|}i}$

$B_k \overset{Pro}{=} Ra_{\overline{n-k|}i} \overset{Ret}{=} L(1+i)^k - Rs_{\overline{k|}i}$

$I_k = i \cdot B_{k-1} = i \cdot Ra_{\overline{n-(k-1)|}i} = R(1 - v^{n-k+1})$

$P_k = R - I_k = Rv^{n-k+1}$

These relationships are captured in a **Level Payment Loan Amortization Table**:

Time	Payment	Interest Paid	Principal Repaid	Balance (Outstanding Principal)
0				$L = B_0 = Ra_{\overline{n|}}$
1	R	$I_1 = R(1 - v^n)$	$P_1 = Rv^n$	$B_1 = Ra_{\overline{n-1|}}$
2	R	$I_2 = R(1 - v^{n-1})$	$P_2 = Rv^{n-1}$	$B_2 = Ra_{\overline{n-2|}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	R	$I_k = R(1 - v^{n+1-k})$	$P_k = Rv^{n+1-k}$	$B_k = Ra_{\overline{n-k|}}$

Remarks about this table:

1. $\{P_1, P_2, \cdots, P_n\}$ is a geometric sequence with common ratio $r = 1 + i$.

2. $L = \sum_{k=1}^{n} P_k = P_1 + P_2 + \cdots + P_n = P_1[1 + (1+i) + \cdots + P_1(1+i)^n] = P_1 s_{\overline{n|}}$

3. We can relate the balance at time k to the balance at time m ($k < m$) as follows:

   $B_k = Ra_{\overline{m-k|}} + B_m v^{m-k}$ (Note that this is a one-step TVM calculation.)

   As written, this equation has a valuation date at time k. Multiplying both sides by $(1+i)^{m-k}$ and rearranging terms gives the time m equation $B_m = B_k(1+i)^{m-k} - Rs_{\overline{m-k|}}$. With $k = 0$, this is the prospective method of determining the balance.

4. As a special case of the previous remark, we can calculate balances at neighboring times in two ways:

$$B_{k+1} = B_k(1+i) - R$$

   or

$$B_{k+1} = B_k - P_{k+1}$$

Module 3 Section 1 Problems:

1. A 10-year loan of 5000 at an annual effective interest rate of 6% is amortized with monthly payments. Determine the amount of the monthly payments.

2. A 20-year loan of 10000 is repaid with quarterly payments of 334.47. Determine the nominal interest rate compounded quarterly charged by the lender.

3. A 30-year mortgage of 200,000 is amortized with monthly payments using a nominal interest rate of 6% compounded monthly.

   (a) Determine the total amount of interest paid on the loan.
   (b) Determine amount of principal repaid during the third 3-year period.
   (c) Determine the amount of interest paid during the 1^{st} year.
   (d) Determine the amount of interest paid during the 30^{th} year.

4. An *n*-year loan of 30000 at 9% interest compounded quarterly is repaid with quarterly payments of 1000 plus an additional final payment.

   (a) Determine the amount of the final payment if it is larger than 1000.
   (b) Determine the amount of the final payment if it is smaller than 1000.

5. A lender charges a nominal interest rate of 3% compounded monthly on a 10-year loan. The amount of principal repaid in the 12^{th} payment is 334.05. Determine the amount borrowed.

6. The lender of a 50,000 loan charges a periodic effective interest rate of 5.06%. The periodic payments are non-level and continue as long as necessary in order to pay off the loan, with the first payment equal to 2000 and subsequent payments increasing by 2% over their previous payments.

   (a) Determine the outstanding principal immediately after the 5^{th} payment.
   (b) Determine the loan balance immediately before the 6^{th} payment.
   (c) Determine the amount of interest paid in the 10^{th} installment.
   (d) Determine the amount of principal repaid in the 10^{th} payment.

7. A 20-year loan at an annual effective interest rate of 4% is repaid with increasing annual payments. The first payment is equal to 1000 and subsequent payments increase by 200 over their previous payments.

   (a) Determine the outstanding principal immediately after the 4^{th} payment.
   (b) Determine the amount of interest paid in the 9^{th} payment.
   (c) Determine the amount of interest paid during the second 4-year period.

Answers to Module 3 Section 1 Problems

1) 55.11

2) 12.16%

3) (a) 231676
   (b) 11216
   (c) 11933
   (d) 457

4) (a) 503.77
   (b) 515.11

5) 45416

6) (a) 52503
   (b) 55160
   (c) 2740
   (d) -350

7) (a) 36523
   (b) 1354
   (c) 5728