Module 4

Addendum to Module 4 Section 6: Macaulay and Modified Convexity

Recall that the **Macaulay duration** of a sequence of future payments is a weighted average of the timing of the payments, where the weight given to the payment at time t is equal to the ratio of the *present value* of the payment at time t to the total *present value* of all payments. The **Macaulay convexity** of a sequence of future payments is a weighted average of *the squares of* the payment times. The weights are the same as above. We get

$$MacD = \frac{\sum t \cdot R_t \cdot v^t}{\sum R_t \cdot v^t}$$

$$MacC = \frac{\sum t^2 \cdot R_t \cdot v^t}{\sum R_t \cdot v^t}$$

Recall that the **modified duration** of a sequence of future payments is defined as

$$ModD = -\frac{P'(i)}{P(i)}$$

where $P(i)$ is the present value, or price, as a function of i. The **modified convexity** of a sequence of future payments is defined as

$$ModC = \frac{P''(i)}{P(i)}$$

Note that $P(i)$ is the denominator in all 4 definitions above. Recall that we showed that $P'(i) = \sum(-t) \cdot R_t \cdot v^{t+1}$, and it follows that, $P''(i) = \sum t(t+1) \cdot R_t \cdot v^{t+2}$. Substituting into the above definitions for *ModD* and *ModC*, and relating back to the definitions of *MacD* and *MacC*, we get the following relationships:

$$ModD = v \cdot MacD$$

$$ModC = v^2 \cdot [MacC + MacD]$$

Important Remarks:

1. When no adjective precedes the word duration, then it means (*Macaulay*) duration, but when no adjective precedes the word convexity, then it means (*modified*) convexity.
2. Just as the duration (Macaulay or modified) of a portfolio is a weighted average of the durations of the assets that comprise the portfolio, with weights being the ratio of the price of the asset to the price of the portfolio, the same is true with convexity (Macaulay or modified).

3. Since Macaulay duration is a weighted average of the timing of the payments, the time unit for Macaulay duration is the same as the time unit for the payments. Since Macaulay convexity is a weighted average of the square of the times of the payments, the time unit for Macaulay convexity is the square of the time unit for the payments. If we have one time unit then we can convert Macaulay values to another time unit. For example, if our time unit is quarters, then we divide the Macaulay duration by 4 to get the Macaulay duration in years, and we divide the Macaulay convexity by $4^2 = 16$ to get the Macaulay convexity in years^2. Notice that we're using Macaulay values to convert from one time unit to another, and not modified values.

4. A fact from calculus is that if we know the values of $P(i_0)$, $P'(i_0)$, and $P''(i_0)$, then given i we can approximate the value of $P(i)$, by using

$$P(i) \approx P(i_0) + P'(i_0) \cdot (i - i_0) + \frac{1}{2} P''(i_0) \cdot (i - i_0)^2$$

   Recall that the price, duration, and convexity are calculated at some given interest rate, i_0. Now consider changing the interest rate by a small amount, Δi, which may be negative. We want to know how this affects the price. We use the approximation above with $i = i_0 + \Delta i$. Solving for $P'(i_0)$ in the definition of *ModD*, and for $P''(i_0)$ in the definition of *ModC*, and using the notation $\Delta P = P(i_0 + \Delta i) - P(i_0)$ for the corresponding change in the price, we get

$$\Delta P \approx -P \cdot ModD \cdot \Delta i + \frac{1}{2} \cdot P \cdot ModC \cdot (\Delta i)^2$$

   where P, *ModD*, and *ModC* are valued at the original interest rate, i_0.

   Therefore, we can think of duration and convexity as measures of the sensitivity of the price to changes in interest rate. Note that if we're not given information on the convexity of the payments, then we can just use the first term (the duration term) in the above formula to determine the approximate change in price.

Addendum to Module 4 Section 6 Problems:

1. You are given a sequence of two payments; 500 at time $t = 2$ and 700 at time $t = 6$.

   (a) Determine the Macaulay duration and Macaulay convexity of the sequence of payments, using a periodic effective interest rate of 3%.

   (b) Determine the modified duration and modified convexity of the sequence of payments, using a periodic effective interest rate of 3%.

2. You are given a sequence of two payments; 500 at the end of 1 year and 700 at the end of 3 years. You are also given an annual effective interest rate of 6.09%. Determine the Macaulay duration, Macaulay convexity, modified duration, and modified convexity of the sequence of payments.

3. You are given the following information on the assets comprising a portfolio:

Asset	Price	Duration (MacD)	Convexity (ModC)
A	2000	6.95	64.05
B	3000	2.92	13.19
C	1000	10.28	129.66

The values in this table have been determined using a periodic effective interest rate of 4%.

(a) Determine the Macaulay convexity of the portfolio.
(b) Using both duration and convexity terms, determine the approximate price of Asset C if the periodic effective interest rate is changed from 4% to 4.05%.
(c) Using duration only, approximate the price of the portfolio if the periodic effective interest rate is changed from 4% to 3.9% for every asset in the portfolio.

Answers to Addendum to Module 4 Section 6 Problems

1) (a) $MacD \doteq 4.217375$ $\quad MacC \doteq 21.739$

(b) $ModD \doteq 4.094539$ $\quad ModC \doteq 24.466373$

2) $MacD \doteq 2.108688$ yrs $\quad MacC \doteq 5.43475$ yrs^2

$ModD \doteq 1.98764$ $\quad ModC \doteq 6.70225$

3) (a) $MacC_{Portfolio} \doteq 48.1087$

(b) $P \doteq \cancel{995.09}$ 995.07

(c) $P \doteq 6031.67$