Mixtures of r.v.'s

Part 1: Discrete mixture of discrete distributions (last)

Part 2: Discrete mixture of continuous r.v.'s

Example: Suppose $X$ is a 80%/20% mixture of $W/Y$ where $W \sim \text{Exp} (\theta = 50)$, $Y \sim \text{Exp} (\theta = 1000)$

Q: $\text{Var} (X) =$?

A: Method 1: (Mix)

$\text{Var} (X) = \frac{\text{E}[X^2]}{\text{mix}} - \left( \frac{\text{E}[X]}{\text{mix}} \right)^2$

$\text{E}[X^2] = 0.8 \cdot \text{E}[W^2] + 0.2 \cdot \text{E}[Y^2]$

$= 0.8 \cdot (2 \cdot 50^2) + 0.2 \cdot (2 \cdot 1000^2) = 4040000$

$\text{E}[X] = 0.8 \cdot \text{E}[W] + 0.2 \cdot \text{E}[Y] = 0.8(50) + 0.2(1000) = 240$

$\therefore \text{Var} (X) = 4040000 - 240^2 = 3469000$

Method 2: (law of total variance)

Let $I =$ "indicator" r.v. ("W" or "Y")

$\text{Var} (X) = \text{E} [\text{Var} (X | I)] + \text{Var} (\text{E}[X | I])$
<table>
<thead>
<tr>
<th>I</th>
<th>$E[X_{11}]$</th>
<th>$Var(X_{11})$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;W&quot;</td>
<td>50</td>
<td>$50^2$</td>
<td>.8</td>
</tr>
<tr>
<td>&quot;Y&quot;</td>
<td>1000</td>
<td>$1000^2$</td>
<td>.2</td>
</tr>
</tbody>
</table>

\[
Var(X) = \left[ \frac{var(E[X_{11}] )}{E[Var(X_{11})]} \right] + \left[ 50^2(.8) + 1000^2(.2) \right]
\]

\[
= 346,400
\]

See Handout for another example
(Next Page)
A portfolio consists of 2500 independent one-year insurance policies. There can be at most one claim made during the year, and for each insured the probability of a claim in the year is 0.02. If a claim is made, the benefit will be exponentially distributed with mean 400. Determine the variance of the total amount of benefits paid in the year.

\[
T = \sum_{i=1}^{2500} Y_i \\
Y \text{ is a } 98\%/2\% \text{ mixture of } 0/X \\
\text{where } X \sim \text{Exp}(\theta = 400)
\]

\[
\text{Var}(T) = 2500 \cdot \text{Var}(Y)
\]

**Method 1:** "Mixing"

\[
\text{Var}(Y) = E[Y^2] - (E[Y])^2
\]

\[
E[Y^2] = .98(0) + .02(400^2) = \frac{6400}{998}
\]

\[
E[Y] = .98(0) + .02(400) = 8
\]

\[
\text{Var}(Y) = \frac{6400}{998} - 8^2 = \frac{1200}{998} = 6.336
\]

\[
\text{Var}(T) = 2500 \left( \frac{1200}{998} \right) = 15,840,000
\]

**Method 2:**

\[
\begin{array}{ccc}
T & | & E[Y] & | & \text{Var}(Y) & | & P(T) \\
\hline
0 & | & 0 & | & 0 & | & .98 \\
400 & | & 400^2 & | & .02 \\
\end{array}
\]

\[
\text{Var}(Y) = \left[ (0^2(.98) + 400^2(.02)) - (0(.98) + 400(.02))^2 \right]
\]

\[
= \left[ (0 + 160000) - (0 + 800)^2 \right] + \left[ \frac{6400}{998} \text{Var}(Y) \right] = 6.336
\]

\[
\text{Var}(T) = 15,840,000
\]
Part 3: Continuous mixture of discrete distribution

Example: \( N | \Lambda = \lambda \sim P(\lambda) \) where \( \Lambda \sim \Gamma(\alpha, \theta) \)
shorten to \( N | \Lambda \sim P(\Lambda) \)

Q: Can we determine the unconditional distribution of \( N \)?

Side Remark: The gamma function (see p. 2 of Tables)
\[
\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt \quad (\alpha > 1)
\]

\[
\begin{align*}
v &= -e^{-t} \\
u &= t^{\alpha-1} \\
0 &= e^{-t} \\
\end{align*}
\]

\[
\begin{align*}
du &= (\alpha-1) t^{\alpha-2} dt \\
dv &= e^{-t} dt
\end{align*}
\]

\[
\begin{align*}
\Gamma(\alpha) &= -e^{-t} \bigg|_0^\infty + \int_0^\infty (\alpha-1) t^{\alpha-2} e^{-t} dt \\
&= (\alpha-1) \cdot \int_0^\infty t^{\alpha-2} e^{-t} dt \\
&= (\alpha-1) \cdot \Gamma(\alpha-1)
\end{align*}
\]

\[
\Gamma(\alpha) = (\alpha-1) \cdot \Gamma(\alpha-1)
\]

Example: \( \frac{\Gamma(\alpha + 5)}{\Gamma(\alpha)} = \frac{(\alpha+1)(\alpha+2)(\alpha+3) \cdots (\alpha+5)}{\Gamma(\alpha)} \)

Generally, \( \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha+1) \cdots (\alpha+k-1) \)