MA56: The \((a, b, 1)\) class of distributions

(aka zero-modified \((a, b, 0)\) distributions)

Idea: Start with a r.v. \(N\) in the \((a, b, 0)\) class

\[
\frac{P_k}{P_{k-1}} = a + \frac{b}{k} \quad \text{start with } P_0
\]

Now change (modify) the \(P_0\) value to \(P_0^M\)

We also must modify all the other \(P_k\) values, but do so in a way that maintains the relative magnitude between the values. Let \(c\) be such that

\[
P_k^M = c \cdot P_k,
\]

Then maintaining the relative magnitude between the values implies that \(P_k^M = c \cdot P_k\) for all other \(k > 1\).

Q: Find a formula for \(c\).

A: Note that \(P_0^M + P_1^M + P_2^M + \cdots = 1\)

\[\Rightarrow P_0^M + c \cdot P_1 + c \cdot P_2 + \cdots = 1\]

\[\Rightarrow P_0^M + c(1 - P_0) = 1\]

\[\therefore c = \frac{1 - P_0^M}{1 - P_0} \quad (= c^M)\]
we have the original \((a, b, 0)\)-distribution, \(N\) and the new zero-modified distribution, \(N^m\)

<table>
<thead>
<tr>
<th>(N)</th>
<th>(P_{r}^m)</th>
<th>(N^m)</th>
<th>(P_{r}^m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(P_0)</td>
<td>0</td>
<td>(P_0^m)</td>
</tr>
<tr>
<td>1</td>
<td>(P_1)</td>
<td>1</td>
<td>(P_1^m = \frac{1-P_0^m}{1-P_0} \cdot P_1)</td>
</tr>
<tr>
<td>2</td>
<td>(P_2)</td>
<td>2</td>
<td>(P_2^m = \frac{1-P_0^m}{1-P_0} \cdot P_2)</td>
</tr>
<tr>
<td>3</td>
<td>(P_3)</td>
<td>3</td>
<td>(P_3^m = \frac{1-P_0^m}{1-P_0} \cdot P_3)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
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<tr>
<td>(\Sigma = 1)</td>
<td>(\Sigma = 1)</td>
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**Remarks:**

1) Note that \(\frac{P_k^m}{P_{k-1}^m} = \frac{P_k}{P_{k-1}} = a + \frac{b}{k}\) starting with \(P_1\)

This is why \(N^m\) is called an \((a, b, 1)\) distribution.

2) The zero-truncated distribution is obtained by setting \(P_0^m = 0\) (Pr\((N^m = 0) = 0\))

3) The zero-modified Poisson distribution is not Poisson. (Likewise for the other \((a, b, 0)\) distributions.)
4) \( E[(N^m)^k] = \frac{1-P_0^m}{1-P_0} E[N] = c^m \cdot E[N]^k \)

\[ \therefore \text{Var}(N^m) = c^m \cdot E[N^2] - (c^m)^2 \cdot (E[N])^2 \]

Ends Module 2: Frequency

See SOA Sample Questions on Frequency posted at the course website.