Module 3: Aggregate (Compound) Distributions

M3SI: IDEA

The total (aggregate) amount of losses per exposure unit can be written as

\[ S = \sum_{1}^{N} X_j \quad \text{with} \quad X_j \sim X \]

and all \( X_j \neq N \) are mutually independent.

We want to know \( \mathbb{E}[S] \) and \( \text{Var}(S) \).

Note: \( S \sim N(\mu, \sigma^2) \)

- \( \mathbb{E}[S] = \mathbb{E}[S|N] = \mathbb{E}[X_1 + X_2 + \ldots + X_N | N] \)

\[ = N \cdot \mathbb{E}[X] \quad (\text{since} \quad \mathbb{E}[X_1] = \mathbb{E}[X_2] = \ldots = \mathbb{E}[X]) \]

Then \( \mathbb{E}[S|N] = N \cdot \mathbb{E}[X] \) can be thought of as a random variable in \( N \).

Note \( \mathbb{E}[X] \) is constant.

:. by double-expectation

\[ \mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S|N]] = \mathbb{E}[N \cdot \mathbb{E}[X]] = \mathbb{E}[X] \cdot \mathbb{E}[N] \]

\[ \therefore \mathbb{E}[S] = \mathbb{E}[N] \cdot \mathbb{E}[X] \]
Var(S) : \[ \text{E}[S|N] = N \cdot \text{E}[X] \] (See above)
Likewise \[ \text{Var}(S|N) = N \cdot \text{Var}(X) \]

By the law of total variance,
\[ \text{Var}(S) = \frac{\text{E}[\text{Var}(S|N)]}{N \cdot \text{Var}(X)} + \text{Var}\left(\frac{\text{E}[S|N]}{N \cdot \text{E}[X]}\right) \]
\[ = \frac{\text{E}[N]}{N \cdot \text{Var}(X)} \cdot \text{Var}(X) + (\text{E}[X])^2 \cdot \text{Var}(N) \]

**M3S2:** Including Deductibles on Each Policy
(possibly other policy modifications)

\[ S = \text{rwr aggregate payments by rws company} \]

\[ S = \sum_{i=1}^{N} Y^L_i \quad \text{or} \quad S = \sum_{i=1}^{N} Y^P_i \]

\[ Y^L = \text{payment per loss r.v.} \quad (E_{Y^L} = (X-d)_+ \quad (E_{Y^P} = X-d | X > d) \]

**Reminder:** Let \( \pi = \text{Pr} \text{(positive pmr)} \)

\[
\begin{array}{c|c|c|c}
N^L & N^P \\
\hline
\text{P} (\lambda) & \text{P}(\lambda \cdot \pi) \\
\text{B} (m, \theta) & \text{B}(m, \theta \cdot \pi) \\
\text{NB}(r, \phi) & \text{NB}(r, \phi \cdot \pi) \\
\end{array}
\]
Example: \( N \sim P(\lambda) \) (X - anything)

Then we have a compound Poisson distribution, \( S \)
\[
S = \sum_i^N X_i
\]

\[
E[S] = \lambda \cdot E[X]
\]

\[
\text{Var}(S) = \lambda \cdot \text{Var}(X) + (E[X])^2 \cdot \lambda = \lambda \cdot (\text{Var}(X) + (E[X])^2)
\]

\[
\therefore \text{Var}(S) = \lambda \cdot E[X^2]
\]

Now suppose \( X \sim \text{Exp}(\theta=1000) \) \( \therefore d = 100 \) on each policy

Also assume \( \lambda = 200 \). (\( \lambda^k \))

Q: \( E[S] = ? \) \( \therefore \text{Var}(S) = ? \)

Method 1: \( S = \sum_i^N Y_i^k \) (This is easier when \( X \sim \text{Exp}(\theta) \)
\( N \sim P(\lambda = 200, \pi) \) since \( Y_i^k \) will also be \( \text{Exp}(\theta) \)

\[
\pi = \Pr(X > d) = \Pr(X > 100) = 1 - e^{-\frac{100}{1000}}
\]

\[
\therefore E[S] = E[N^P] \cdot E[Y_i^k] = \lambda^P \cdot E[Y_i^k]
\]

\[
= (200e^{-1}) \cdot (1000)
\]

\[
\text{Var}(S) = \lambda^P \cdot E[Y_i^k]^2
\]

\[
= (200e^{-1}) \cdot (2 \cdot 1000^2)
\]

Method 2: \( S = \sum_i^N Y_i^k \) \( N^k \sim P(\lambda^k = 200) \)
\[ E[S] = X^t \cdot E[Y^t] \quad \text{Var}(S) = X^t \cdot E[(Y^t)^2] \]

\[ X \sim \text{Exp}(\theta = 1000) \quad d = 100 \]

\[ Y^t = (X - 100)^+ = \begin{cases} 0 & \text{if } X < 100 \\ X - 100 & \text{if } X \geq 100 \end{cases} \]

Long way is to integrate to get 1st & 2nd moments
It’s easier to use the fact:

\[ E[(Y^t)^k] = E[(Y^p)^k] \cdot Pr(X > d) \]

Then \[ E[S] = X^t \cdot E[(Y^p)^k] \cdot \frac{1}{\pi} = \pi^t \cdot E[(Y^p)^k] \]

\[ \text{Var}(S) = X^t \cdot E[(Y^p)^2] \cdot \frac{1}{\pi} = \pi^t \cdot E[(Y^p)^2] \]

Summary: \[ E[S] = 200,000 \quad \text{e}^{-1} \]

\[ \text{Var}(S) = 400,000,000 \quad \text{e}^{-1} \]

Then \[ \tau_S = 20,000 \quad \text{e}^{-0.05} \]

\[ \text{Q: } Pr(S < 200,000) \]

\[ A: \quad = Pr(S_{\text{ND}} < \frac{200,000 - 200,000 \text{e}^{-1}}{20000 \text{e}^{-0.05}} = 1) \]

Standard normal distribution “Z”

\[ \therefore Pr(S < 200,000) = Pr(S_{\text{ND}} < 1) = .84 \]