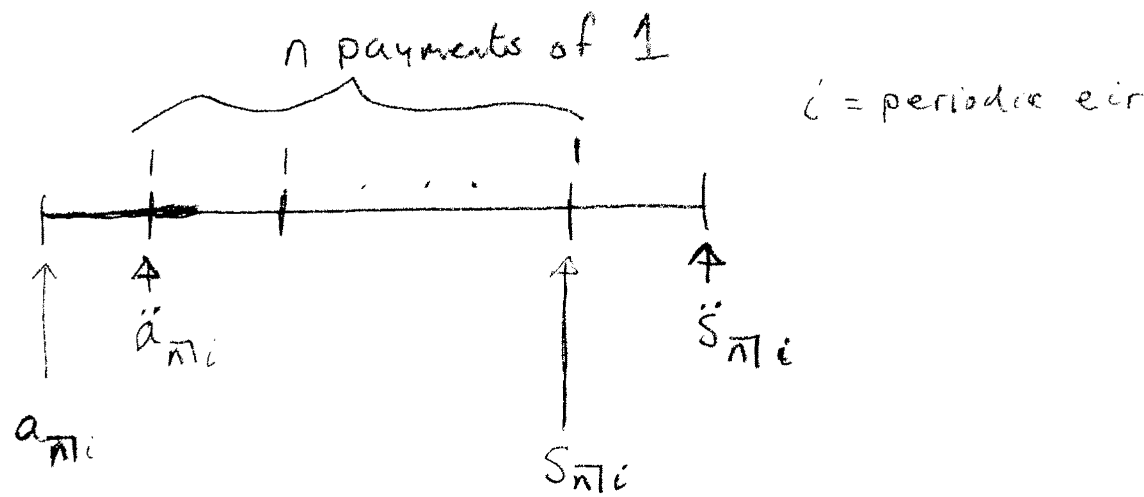
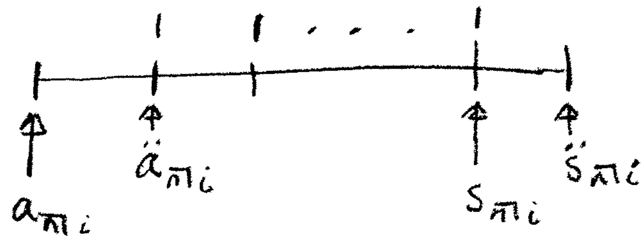
**Module 2**

## Section 1: Annuities – Definition, Terminology, and Notation

An **annuity** is a sequence of periodic payments. The annuities on Exam FM/2 are called **certain annuities** because we assume the payments are certain to be paid, as opposed to the annuities on Exam MLC in which the payments will depend on the occurrence of some event. The **annuity start date** is at the beginning of the first period and the **annuity end date** is at the end of the last period. An **ordinary annuity** is an annuity in which the payments are thought of as taking place at the end of each period. An ordinary annuity is also called an **annuity immediate**. An **annuity due** is an annuity in which payments are thought of as taking place at the beginning of each period. All financial calculations have a **valuation date**. The **value** of an annuity at a given valuation date is the single sum value at the valuation date in which one is indifferent to receiving instead of receiving the periodic payments. The **present value** of an annuity is the value of the annuity at the annuity start date, and the **future** (or **accumulated**) **value** of an annuity is the value of the annuity at the annuity end date. If the valuation date is between the annuity start and end dates, then the value is called a **(current) value**. If the valuation date is before the annuity start date, then the annuity is called a **deferred annuity**. An annuity is called **level** if the payments are equal. We first focus on level annuities, and just as with accumulation functions, the development of formulas is based on a "per dollar of payment" basis. So we start with a level annuity with payments of 1, which I'll call a **basic level annuity**. Of course, the value of an annuity will depend on the interest rate. Unless (on the rare occasion) told otherwise, we determine values of annuities using periodic effective *interest* rates (eir's). In the timeline below, we introduce the notation for the value of a basic level annuity at each of the valuation dates shown.



Summary:




$a_{\overline{n}|i}$ is read "a, angle n, at *i*". It represents the value, one payment period before the first payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of an (basic level) annuity immediate. If there is no ambiguity, then the "*i* " in the notation is often omitted.

$\ddot{a}_{\overline{n}|i}$ is read "a, double dot, angle n, at *i*". It represents the value, immediately before the first payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of an (basic level) annuity due. If there is no ambiguity, then the "*i* " in the notation is often omitted.

$s_{\overline{n}|i}$ is read "s, angle n, at *i*". It represents the value, immediately after the last payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the accumulated (or future) value of an (basic level) annuity immediate. If there is no ambiguity, then the "*i* " in the notation is often omitted.

$\ddot{s}_{\overline{n}|i}$ is read "s, double dot, angle n, at *i*". It represents the value, one payment period after the last payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the accumulated (or future) value of an (basic level) annuity due. If there is no ambiguity, then the "*i* " in the notation is often omitted.

Some Elementary Relationships:

| | | |
|---|---|---|
| $a_{\overline{n}\vert} = \ddot{a}_{\overline{n}\vert} \cdot v$ | (or equivalently) | $\ddot{a}_{\overline{n}\vert} = a_{\overline{n}\vert} \cdot (1+i)$ |
| $s_{\overline{n}\vert} = \ddot{s}_{\overline{n}\vert} \cdot v$ | (or equivalently) | $\ddot{s}_{\overline{n}\vert} = s_{\overline{n}\vert} \cdot (1+i)$ |
| $a_{\overline{n}\vert} = s_{\overline{n}\vert} \cdot v^n$ | (or equivalently) | $s_{\overline{n}\vert} = a_{\overline{n}\vert} \cdot (1+i)^n$ |
| $\ddot{a}_{\overline{n}\vert} = \ddot{s}_{\overline{n}\vert} \cdot v^n$ | (or equivalently) | $\ddot{s}_{\overline{n}\vert} = \ddot{a}_{\overline{n}\vert} \cdot (1+i)^n$ |

A **perpetuity** is an annuity in which the payments continue forever.

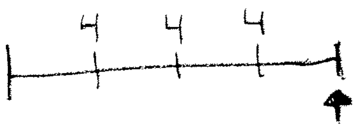
$a_{\overline{\infty}|i}$ represents the value, one payment period before the first payment, of an infinite number of periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of a (basic level) perpetuity immediate. Likewise $\ddot{a}_{\overline{\infty}|i}$ represents the value, immediately before the first payment, of an infinite number of periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of a (basic level) perpetuity due.

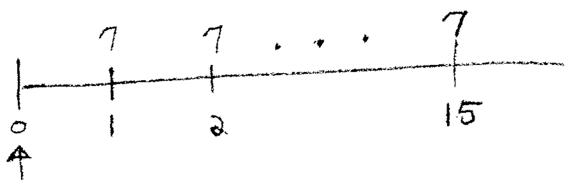
Note that $a_{\overline{\infty}|} = \ddot{a}_{\overline{\infty}|} \cdot v$ (or equivalently) $\ddot{a}_{\overline{\infty}|} = a_{\overline{\infty}|} \cdot (1+i)$

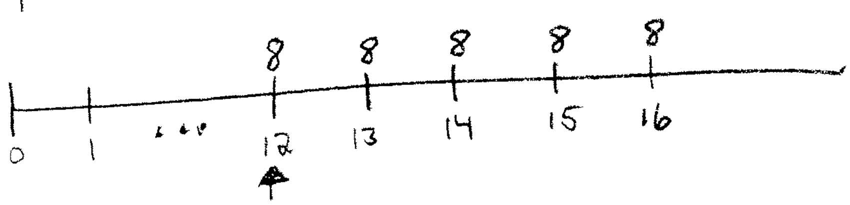
**Module 2 Section 1 Problems:**

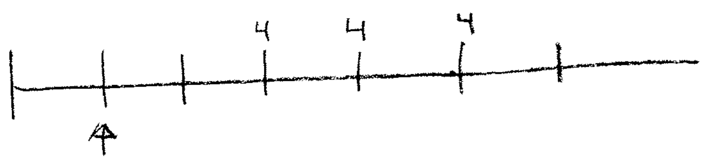
For Numbers 1-10, give an expression for the value, at the given valuation date, of the cash flow shown. The valuation date is marked with a vertical arrow. You cannot determine a numeric value for the expression until you are given an interest rate to use. We'll do that in the next section.

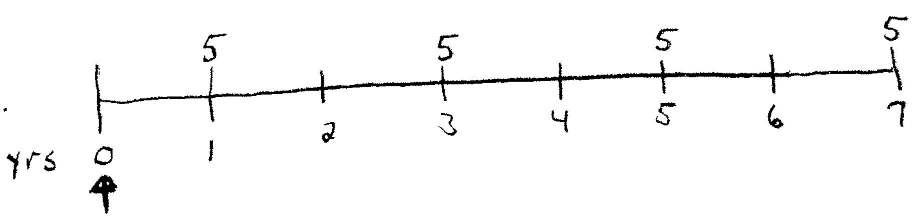
Note: You will not see these types of problems on Exam FM/2, but it is essential that you know how to do these problems to continue.

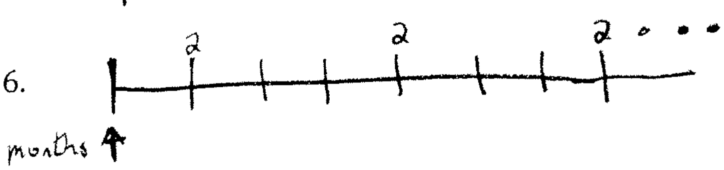
1. 

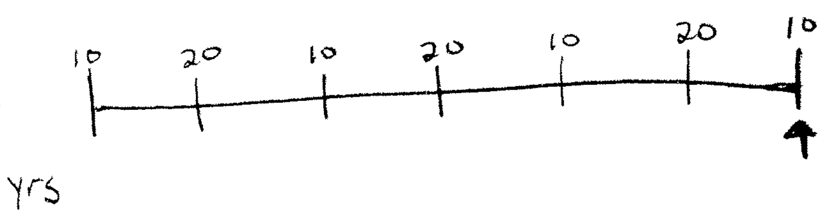

2. 

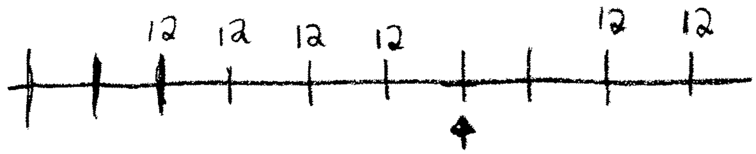

3. 

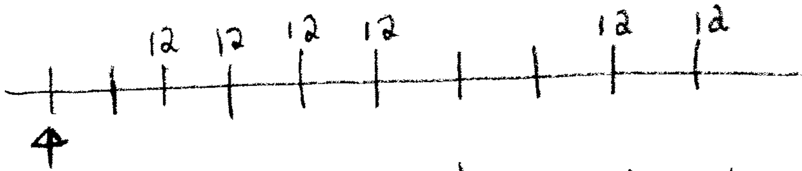

4. 

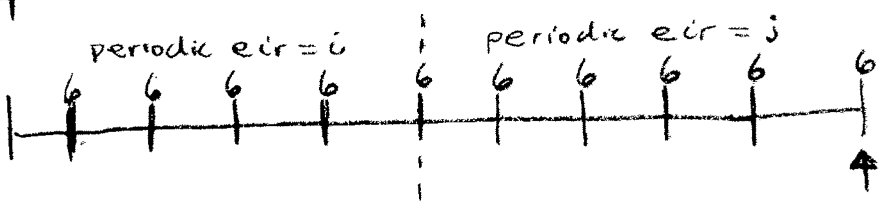

5. 


6. 


7. 

8. 


9. 


10. 


For Numbers 11-14, draw the timeline that corresponds to the given expression.

11. $3s_{\overline{6}|i} + 4\ddot{s}_{\overline{2}|j}$ where $i$ is an annual effective interest rate and $j$ is a biannual effective interest rate.

12. $7a_{\overline{7}|} - 7\ddot{a}_{\overline{3}|}v^3$

13. $5a_{\overline{6}|} + 100v^6$

14. $10(1+i)^5 + 3s_{\overline{5}|i}$

# Answers to Module 2 Section 1 Problems

(For #s 1–10, There are multiple correct answers. The ones below are some that are more intuitive to me.)

1) $4\ddot{s}_{\overline{3}|}$

2) $7a_{\overline{15}|}$

3) $8\ddot{a}_{\overline{5}|}$

4) $4a_{\overline{3}|}\cdot v$ or $4\ddot{a}_{\overline{3}|}v^2$

5) $5\ddot{a}_{\overline{4}|j}\cdot v_j^{1/2} = 5\ddot{a}_{\overline{4}|j}\cdot v_i$ $\quad i = aeir;\ j = baeir$

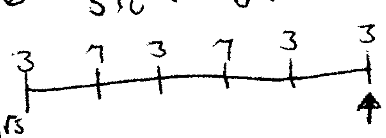
6) $2\ddot{a}_{\overline{\infty}|j}\cdot v_j^{1/3} = 2\ddot{a}_{\overline{\infty}|j}\cdot v_i$ $\quad i = meir;\ j = qeir$

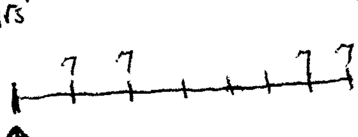
7) $20s_{\overline{7}|i} - 10s_{\overline{4}|j}$ $\quad i = aeir;\ j = baeir$

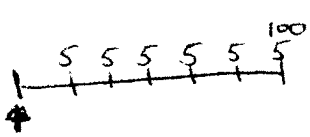
8) $12\ddot{s}_{\overline{4}|} + 12\ddot{a}_{\overline{2}|}\cdot v^2$

9) $12a_{\overline{8}|}\cdot v - 12a_{\overline{2}|}\cdot v^5$

10) $6s_{\overline{5}|i}(1+j)^5 + 6s_{\overline{5}|j}$

11) 


12) 


13) 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