Section 7: Advanced Upper *m* and Continuous Annuities

Although technically on the syllabus, the material in this section is very rarely tested, and when it is there are methods to solve problems that bypass the material in this section. The examples at the end of the section will illustrate this point. We will present the material here for completeness, but if pressed for time, the material in this section can be safely omitted from your studies.

Advanced Upper *m* Annuities

By advanced upper *m* notation, we mean combining upper *m* annuity notation with arithmetically increasing/decreasing annuity notation to get symbols like

$$(Ia)^{(m)}_{\overline{n}|i} \quad (I\ddot{a})^{(m)}_{\overline{n}|i} \quad (Is)^{(m)}_{\overline{n}|i} \quad (I\ddot{s})^{(m)}_{\overline{n}|i} \quad (Da)^{(m)}_{\overline{n}|i} \quad (D\ddot{a})^{(m)}_{\overline{n}|i} \quad (Ds)^{(m)}_{\overline{n}|i} \quad (D\ddot{s})^{(m)}_{\overline{n}|i}$$

$$(I^{(m)}a)^{(m)}_{\overline{n}|i} \quad (I^{(m)}\ddot{a})^{(m)}_{\overline{n}|i} \quad (I^{(m)}s)^{(m)}_{\overline{n}|i} \quad (I^{(m)}\ddot{s})^{(m)}_{\overline{n}|i} \quad (D^{(m)}a)^{(m)}_{\overline{n}|i} \quad (D^{(m)}\ddot{a})^{(m)}_{\overline{n}|i}$$

$$(D^{(m)}s)^{(m)}_{\overline{n}|i} \quad (D^{(m)}\ddot{s})^{(m)}_{\overline{n}|i}$$

Instead of examining these symbols separately and deriving VEP and CRF's for each of the symbols, let's discuss the notation. Just as with basic upper *m* notation, the (m) above the symbol $\overline{n}|i$ represents the fact that there are *m* payments per (interest conversion) period. In order to simplify matters, but without loss of generality, let's assume that *i* is an aeir, and so the period is in years. Now, the symbol *I* represents the fact that the payments are increasing each year, starting with a payment of 1 for the first year, then 2 for the second year, and so on. Putting both these symbols together then implies there are *m* payment during the first year totaling 1, *m* payments during the second year totaling 2, and so on. That is, the payments during the first year are level at $1/m$ each, the payments during the second year are level at $2/m$ each, and so on. Then a symbol like $(Ia)^{(m)}_{\overline{n}|i}$ represents the present value of this payment stream with a valuation date being one payment period before the first payment. The CRF's for these advanced upper *m* symbols are obtained from the CRF's for the corresponding basic upper *m* symbols by replacing the periodic rate in the denominator by it's equivalent "upper *m*" nominal rate. For example, since $(D\ddot{a})_{\overline{n}|} = \frac{n-a_{\overline{n}|}}{d}$ then $(D\ddot{a})^{(m)}_{\overline{n}|i} = \frac{n-a_{\overline{n}|}}{d^{(m)}}$ Likewise for the others CRF's.

Now consider placing the (m) next to the *I* or *D*, getting $I^{(m)}$ or $D^{(m)}$. This implies that the increase, or decrease, takes place *m* times per year also. The upper *m* next to the *I* or *D* is always accompanied by the upper *m* over the $\overline{n}|i$. That is, you will not see symbols such as $(I^{(2)}s)^{(4)}_{\overline{n}|i}$. For these "double upper *m*"annuities, the payments are made *m* times per year and the increase is made *m* times per year. That is, the increase occurs with each payment. For the basic increasing double upper m annuity, the first payment is $\frac{1}{m^2}$, the second payment is $\frac{2}{m^2}$, and so on.

The CRF's for these double upper m annuities are obtained from the CRF's for the corresponding advanced upper m annuities above by replacing the annuity symbol in the numerator by its corresponding upper m annuity symbol. For example, since $(Is)_{\overline{n}|i}^{(m)} = \frac{\ddot{s}_{\overline{n}|}-n}{i^{(m)}}$ then $(I^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{\ddot{s}_{\overline{n}|}^{(m)}-n}{i^{(m)}}$. Similarly, since $(D\ddot{a})_{\overline{n}|i}^{(m)} = \frac{n-a_{\overline{n}|}}{d^{(m)}}$ then $(D^{(m)}\ddot{a})_{\overline{n}|i}^{(m)} = \frac{n-a_{\overline{n}|}^{(m)}}{d^{(m)}}$. Likewise for the other CRF's. Although these are the CRF's for double upper m annuities, there is an easier way to determine the value for these symbols. If we factor out the first term in the VEP formulas, then the second factor will just be a basic increasing or decreasing annuity with mn payments, although we must adjust the periodic effective interest rate to match the period of the payments. For example, for the two formulas above, we have $(I^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{1}{m^2}(Is)_{\overline{mn}|j}$ and $(D^{(m)}\ddot{a})_{\overline{n}|i}^{(m)} = \frac{1}{m^2}(D\ddot{a})_{\overline{mn}|j}$ where j is the periodic effective interest rate and where there are m periods per year.

Advanced Continuous Annuities

The annuities discussed thus far are called discrete annuities since the payments are made at discrete times during the year. We get advanced continuous annuities by taking a limiting process, as $m \to \infty$, of advanced upper m annuities. Note that as $m \to \infty$, then $i^{(m)}$ and $d^{(m)} \to \delta$, and with annuity symbols, upper ∞ is replaced by using a "bar". This allows us to derive advanced continuous annuity CRF's from advanced upper m CRF's. For example,

$$(Ds)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - s_{\overline{n}|}}{i^{(m)}} \quad \Longrightarrow \quad (D\bar{s})_{\overline{n}|i} = \lim_{m\to\infty}(Ds)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - s_{\overline{n}|}}{\delta}$$

$$(D^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - s_{\overline{n}|}^{(m)}}{i^{(m)}} \quad \Longrightarrow \quad (\bar{D}\bar{s})_{\overline{n}|i} = \lim_{m\to\infty}(D^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - \bar{s}_{\overline{n}|}}{\delta}$$

Let's discuss the difference between the continuous annuities that are being valued with the two equations on the right hand side of the implications above. The continuous annuity corresponding to the symbols with a "bar" over the a or s, *and* a "bar" over the I or D, is an annuity in which the payments are made continuously, and the increase or decrease is made continuously. For example, the symbol $(\bar{I}\bar{a})_{\overline{n}|i}$ represents the present value of an annuity in which payments are made continuously and the payment rate at time t is equal to t. Technically, we say that the payment rate function is $f(t) = t$, which means we can approximate the amount the annuity pays from time t to time $t + \Delta t$ by the expression $t \cdot \Delta t$. Thus we have $(\bar{I}\bar{a})_{\overline{n}|i} \overset{VEP}{=} \int_0^n t \cdot v^t \, dt$ and $(\bar{I}\bar{s})_{\overline{n}|i} \overset{VEP}{=} \int_0^n t \cdot e^{\delta(n-t)} \, dt$. Notice we can factor the exponential in the integrand of the last integral as $e^{\delta n} \cdot e^{-\delta t} = e^{\delta n} \cdot v^t$. Since $e^{\delta n} = (1+i)^n$ is constant with respect to t, we can factor it out. The result is that $(\bar{I}\bar{s})_{\overline{n|}i} = (1+i)^n \cdot \int_0^n t \cdot v^t \, dt = (1+i)^n \cdot (\bar{I}\bar{a})_{\overline{n|}i}$ which is what we already know; namely, that $AV = PV \cdot (1+i)^n$.

The continuous annuity corresponding to the symbols with a "bar" over the a or s, but no "bar" over the I or D, is an annuity in which the payments are made continuously, but the increase or decrease is made discretely. For example, the symbol $(I\bar{a})_{\overline{n|}i}$ represents the present value of an annuity in which payments are made continuously and total 1 during the first year, 2 during the second year, etc. So the payment rate function is the piecewise-defined function

$$f(t) = \begin{cases} 1 & \text{if } 0 < t \leq 1 \\ 2 & \text{if } 1 < t \leq 2 \\ & \vdots \\ n & \text{if } n-1 < t < n \end{cases}$$

Since the payment rate function is defined as such, the VEP (integral) expression for such annuities will actually be a sum of n integrals. For example, we have

$$(I\bar{a})_{\overline{n|}i} \overset{VEP}{=} \int_0^1 1 \cdot v^t \, dt + \int_1^2 2 \cdot v^t \, dt + \cdots + \int_{n-1}^n n \cdot v^t \, dt$$

General Payment Rate Functions

We generalize the payment rate function to a general $f(t)$ to get present values and accumulated values of general continuous annuities. There is no actuarial notation in this general form, and so we just write

$$PV \overset{VEP}{=} \int_0^n f(t) \cdot v^t \, dt$$

$$AV \overset{VEP}{=} \int_0^n f(t) \cdot e^{\delta(n-t)} \, dt$$

General Payment Rate Functions and General Force of Interest

Finally, the most general situation is if we have a general payment rate function $f(t)$ and a general force of interest δ_t (as opposed to a constant force of interest δ). Then

$$PV \overset{VEP}{=} \int_0^n f(t) \cdot e^{\int_t^0 \delta_s ds} dt$$

$$AV \overset{VEP}{=} \int_0^n f(t) \cdot e^{\int_t^n \delta_s ds} dt$$

Module 2 Section 7 Problems:

1. Determine the present value of a 10-year annuity immediate with level monthly payments of 2 for the first year, level monthly payments of 4 for the second year, level monthly payments of 6 for the third year, and so on. Use an annual effective interest rate of 5%.

2. Determine the accumulated value of a 5-year annuity due with level semiannual payments of 25 for the first year, level semiannual payments of 20 for the second year, level semiannual payments of 15 for the third year, and so on. Use an annual effective interest rate of 6%.

3. Determine the accumulated value of a 10-year annuity immediate with level monthly payments of 9 for the first year, level monthly payments of 11 for the second year, level monthly payments of 13 for the third year, and so on. Use an annual effective interest rate of 5%.

4. Determine the accumulated value of a 9-year annuity immediate with semiannual payments that start at 10 and increase each semiannual period by 4. Use an annual effective interest rate of 8.16%.

5. Determine the present value of a 15-year continuous annuity in which the payment rate during the first year is 1, the payment rate during the second year is 2, and so on. Use a force of interest of 3%.

6. Determine the accumulated value of a 20-year decreasing continuous annuity in which the payment rate at time t is $20 - t$. Use a force of interest of 4%.

7. Determine the accumulated value of a 3-year continuous annuity with payment rate function $f(t) = 9t^2$ where the force of interest at time t is given by $\delta_t = \frac{t^2}{9}$.

Answers to Module 2 Section 7 Problems

1) 966.4356

2) 183.5394

3) 2654.764

4) 1020.995

5) 89.891

6) 301.281

7) $81(e-1)$