Section 5: Duration

The **Macaulay duration** (or just **duration**) of a sequence of future payments is a measure of the timing of the payments. It is a weighted average of the timing of the payments, where the weight given to the payment time *t* is equal to the ratio of the *present value* of the payment at time *t* to the total *present value* of all payments. Since the weights are ratios of present values, then the duration will depend on the interest rate used to discount the payments. I.e., the duration is a function of the interest rate.

If payments of X and Y are made at times k and n, respectively, then the (Macaulay) duration of this cash flow is $MacD = \frac{Xv^k}{Xv^k+Yv^n} \cdot k + \frac{Yv^n}{Xv^k+Yv^n} \cdot n$, which can be simplified to $MacD = \frac{kXv^k+nYv^n}{Xv^k+Yv^n}$. This last formula can be generalized to an arbitrary number of payments as follows, letting R_t denote the amount of the payment at time t:

$$MacD = \frac{\sum t \cdot R_t \cdot v^t}{\sum R_t \cdot v^t}$$

Note that the denominator is just the present value of the payments. E.g., if the payments are the coupons and redemption value of a bond, then the denominator is just the price of the bond. Sometimes instead of price, we say the **present value function**, which we denote by $P(i)$, emphasizing the fact that it depends on the interest rate.

Note that since $P(i) = \sum R_t \cdot (1+i)^{-t}$, then using the power rule for derivatives, we have $P'(i) = \sum(-t) \cdot R_t \cdot (1+i)^{-t-1} = -\sum t \cdot R_t \cdot v^{t+1}$. Then, factoring out a v from the summand, we have $P'(i) = -v \sum t \cdot R_t \cdot v^t$. The summation in this last expression is the numerator of the defining expression for *MacD*. This fact is one justification the following definition.

The **modified duration** (or **volatility**) of a sequence of future payments is

$$ModD = -\frac{P'(i)}{P(i)} = v \cdot MacD$$

Remark: Since the price is a decreasing function of the interest rate, the negative sign in the above formula ensures that *ModD* is positive. Also, for small changes in the interest rate, Δi, then $P(i) \cdot ModD \cdot \Delta i$ will approximate the corresponding change in the price. Therefore, *ModD* is a measure of how sensitive the present value of the future cash flow is to small changes in the interest rate.

Module 4 Section 5 Problems:

1. Determine the duration of a 10-year zero coupon bond, redeemable at 1000, using an annual effective interest rate of 5%.

   Note: Think about it. Look back at the definition of duration, and you should be able to answer this problem without putting pencil to paper.

2. You are given a sequence of 2 payments; 1000 at the end of 2 years and 3000 at the end of 6 years.

   (a) Determine the modified duration, using an annual effective interest rate of 4%.

   (b) Determine the time value calculated by the method of equated time. The method of equated time is the special case of duration, using a 0% interest rate.

3. Determine the duration of 20-year 6% annual coupon bonds using

   (a) an annual effective interest rate of 8%.

   (b) an annual effective interest rate of 6%.

   Note: Assume the bond is redeemable at par. The face value is not given, and in fact, we will show that for bonds redeemable at par, the duration is independent of the face value. Therefore you may use any face value you want.

4. Determine the duration of a perpetuity-immediate with annual payments of K, using an annual effective interest rate of 8%.

5. Determine the modified duration (in years) of a perpetuity-due with monthly payments of K, using a nominal interest rate of 6% compounded monthly.

6. A stream of periodic payments has a present value of 5000 and a modified duration of 5, both using a periodic effective interest rate of 5%. Approximate the present value of the payments using a periodic effective interest rate of 5.02%.

Answers to Module 4 Section 5 Problems

1) 10

2) (a) 4.69

   (b) 5

3) (a) 11.231

   (b) 12.158

4) 13.5

5) 16.58

6) 4995