For 1997: \[ 100000(1+x) - 8000(1 + .75x) = B \] \hspace{1cm} (1)

For 1998: \[ 1 + x = \frac{103992}{B} \] \hspace{1cm} (2)

(1) \[ \Rightarrow \quad 92000 + 94000x = B \]

(2) \[ \Rightarrow \quad B = \frac{103992}{1+x} \]

\[ \therefore \quad 92000 + 94000x = \frac{103992}{1+x} \]

\[ \Rightarrow (1+x)(92000 + 94000x) = 103992 \]

\[ \therefore \quad 94000x^2 + 186000x - 11992 = 0 \]

\[ \Rightarrow \quad x = \frac{-186000 \pm \sqrt{(186000)^2 + 4(94000)(11992)}}{2(94000)} \]

\[ = .0625 \]
PV = X

\[ \dot{i}' = \text{real rate of return} = \begin{cases} \dot{i}' = 0.063 & 0 \leq t \leq 6 \\ \frac{1.063}{1.012} - 1 & t > 6 \end{cases} \]

\[ 1 + \dot{i}' = \frac{1 + \dot{i}'}{1 + r} \]

\[ X = PV = 50 \frac{a_{\ddot{i}'}}{12} \left( \frac{1.063}{1.012} - 1 \right) \cdot 2^{\dot{i}'}, 0.063 \]

\[ = 306.48 \]
The amount of interest credited during 1993 equals the balance at the end of 1992 times the credited interest rate for 1993.

\[ 28.40 = 100(1.1)(1.1)(1+t)(0.08) + 100(1.12)(1.05)(0.1) + 100(1.08)(t - 0.02) \]

\[ \Rightarrow t = 0.0775 \]
Using TW, the annual yield for the first 6 months is determined by:
\[
(1+i)^{1/2} = \frac{40}{50} \cdot \frac{80}{60} \cdot \frac{157.50}{160} = 1.05
\]

\[\Rightarrow i = 10.25\%\]

\[\therefore\] the TW annual effective yield for the entire year is 10.25%.

\[\therefore 1.1025 = \frac{40}{50} \cdot \frac{80}{60} \cdot \frac{175}{160} \cdot \frac{x}{250}\]

\[\Rightarrow x = 236.25\]
Chuck needs \(200(1.04)^{10} = 296.05\) in 10 years.

Chuck needs 296.05 at time \(t=10\). His investments earn 10% aear. We don’t use the real rate of return when accumulating his deposits, since we’ve already accounted for inflation when determining the amount Chuck needs at time \(t=10\).

The equation we solve is:

\[
296.05 = 20\, S_{67.1} \,(1.1)^5 + X\, S_{37.1} \,(1.1)^5
\]

we could have used \(S\)'s if we wanted.

\[
\Rightarrow X = 8.9
\]
\[ i_{tw} = 0 \implies 1 + 0 = \frac{12}{10} \cdot \frac{x}{12+x} \implies 1 = \frac{1.2x}{12+x} \]

\[ \implies 12 + x = 1.2x \implies x = 60 \]

\[ i_{dw} = y \implies 10(1+y) + x(1+\frac{1}{2}y) = x \]

\[ \implies 70 + 40y = 60 \]

\[ \implies y = -0.25 \]
\[ \frac{125}{125 - x} \]
\[ \frac{110 + 2x}{110 + 2x} \]
\[ 125 \]

\[ \text{Account } K : \]

\[ i_{DW} = i \Rightarrow 100 (1+i) - x(1+1.5i) + 2x(1+1.25i) = 125 \]

\[ \text{Account } L : \]

\[ i_{TW} = i \Rightarrow 1+i = \frac{125}{100} \cdot \frac{105.8}{125-x} \Rightarrow (1+i)(125-x) = 132.25 \]

Solve (1) and (2) simultaneously. (Many ways!)

Let's solve for \( x \) in (1) and substitute into (2).

From (1): \[ 100 + 100i \cdot \frac{-x - .5xi}{125-x} + 2x + .5xi = 125 \]

\[ \Rightarrow x = 25 - 100i \]

by (2), \( (1+i)(125-(25-100i)) = 132.25 \)

\[ \Rightarrow (1+i)(100+100i) = 132.25 \]

\[ \Rightarrow 100(1+i)^2 = 132.25 \Rightarrow i = 15\% \]
\[ A = \text{Boy Amount} = 75 \]
\[ B = \text{E04 Amount} = 60 \]

\[ \text{Withdraws Deposits} \]
\[
\begin{array}{cccccccccc}
& 0 & \frac{1}{12} & \frac{3}{12} & \frac{4}{12} & \frac{5}{12} & \frac{6}{12} & \frac{7}{12} & \frac{8}{12} & \frac{9}{12} & \frac{10}{12} \\
\hline
\text{Withdraw} & 25 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
\text{Deposits} & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
\end{array}
\]

\[ i_{DW} = i \implies \]

\[ 75 (1 + i) + 10 \left(1 + \frac{10}{12} i\right) + 10 \left(1 + \frac{10}{12} i\right) + \ldots + 10 \]

\[ -5 \left(1 + \frac{10}{12} i\right) - 25 \left(1 + \frac{6}{12} i\right) - 35 \left(1 + \frac{2}{12} i\right) = 60 \]

\[ -5 \left(1 + \frac{10}{12} i\right) - 25 \left(1 + \frac{6}{12} i\right) - 35 \left(1 + \frac{2}{12} i\right) = 60 \]

\[ \frac{75i}{12} + 10 \left(1 + \frac{10}{12} i\right) \left(11 + 10 + 9 + \ldots + 1\right) = 60 \]

\[ \frac{75i}{12} + 10 \left(1 + \frac{10}{12} i\right) \left(\frac{n(n+1)}{2}\right) = 60 \]

\[ \frac{75i}{12} + 10 \left(1 + \frac{10}{12} i\right) \left(\frac{11(12)}{2}\right) = 60 \]

\[ \frac{75i}{12} + 120 + 55i = 145 \]

\[ \frac{770}{12} i = 60 \]

\[ \implies i = 11.90 \]


\[ P = 1000 \times (1.095)^2 \times (1.096) \]

\[ Q = 1000 \times (1.0835) \times (1.086) \times (1.0885) \]

\[ R = 1000 \times (1.095) \times (1.1)^2 \]

\[ \therefore R > P > Q \]
\[
\frac{X = \text{Mac} D =}{100 (2) v^2 + 100 (3) v^3 + 1000 (2) v^3}{100 v + 100 v^2 + 100 v^3 + 1000 v^3}
\]
\[
\frac{100 (Ia)^3_1 + 3000 v^3}{100 a^3_1 + 1000 v^3} = 2.70
\]
Total Portfolio Price
\[ = 980 + 1015 + 1000 = 2995 \]

\[ \text{MacD}_{\text{Portfolio}} = \left( \frac{980}{2995} \right) (21.46) + \left( \frac{1015}{2995} \right) (12.35) + \left( \frac{1000}{2995} \right) (16.67) \]

\[ \approx 16.77 \text{ years} \]
\[
(1.095)^2 = (1.085)(1+c) \Rightarrow c = 10.5\% 
\]
let $F_1 =$ face amount of the 1-year bond and $F_2 =$ face amount of the 2-year bond

Assets

<table>
<thead>
<tr>
<th>Year</th>
<th>1.06 $F_2$</th>
<th>$F_1$</th>
<th>0.96 $F_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.96 $F_2$</td>
</tr>
<tr>
<td>1</td>
<td>1.04 $F_1$</td>
<td>0.96 $F_2$</td>
<td>1.04 $F_1$</td>
</tr>
<tr>
<td>2</td>
<td>1.06 $F_2$</td>
<td>1</td>
<td>1.06 $F_2$</td>
</tr>
</tbody>
</table>

Liabilities

- $10000$
- $10000$

Exact Matching $\Rightarrow$

\[
\begin{align*}
1.04 F_1 + 0.6 F_2 &= 10000 \\
1.06 F_2 &= 10000 \\
F_1 &= 9071.1176 \\
F_2 &= 9433.9623
\end{align*}
\]

Prices:

\[
\begin{align*}
P_1 &= 1.04 F_1 \cdot a_{\overline{7.05}} + F_1 \cdot \overline{v_{0.05}} = 1.04 F_1 \cdot 2.05 = 8984.73 \\
P_2 &= 1.04 F_2 \cdot a_{\overline{7.05}} + F_2 \cdot \overline{v_{0.05}} = 0.6 F_2 \cdot 2.05 + 1.06 F_2 \cdot \overline{v_{0.05}} = 9609.38
\end{align*}
\]

so total cost $X = P_1 + P_2 = 18594$

Notes: 1) In reality, the 1-year bond has a fixed face value, $C_1$, and we are purchasing an unknown number, $N_1$, of 1-year bonds. In the manual we showed this is equivalent to purchasing one 1-year bond with an unknown face value $F_1$ ($F_1 = C_1 \cdot N_1$). Likewise for the 2-year bond.

2) $X = P_1 + P_2 = 1.04 F_1 \cdot \overline{v} + 0.6 F_2 \cdot \overline{v} + 1.06 F_2 \cdot \overline{v}^2$

The $\overline{v}$ in the first term uses the yield on the 1-year bond whereas the $\overline{v}$'s in the last two terms use the yield on the 2-year bond. Since the yields are the same in this problem, we get $X = (1.04 F_1 + 0.6 F_2) \cdot \overline{v} + 1.06 F_2 \cdot \overline{v}^2 = 10000 \overline{v}_{0.05} + 10000 \overline{v}^2_{0.05}$; we didn't need to find $F_1$ and $F_2$, since the yields were equal.
Withdrawals
Deposits

\[ \begin{array}{llll}
0 & \frac{1}{12} & \frac{6}{12} & \frac{8}{12} & 1 \\
\hline
1000 & 200 & 500 & \\
\end{array} \]

\[ i_{DW} = i \]

\[ 1000 (1 + \frac{7}{12} i) + 1000 (1 + \frac{6}{12} i) - 200 (1 + \frac{6}{12} i) - 500 (1 + \frac{4}{12} i) = 1560 \]

\[ \Rightarrow i = 18.57\% \]

**Alternative Solution**

\[ A = 1000 \]
\[ B = 1560 \]
\[ C = \text{net total contributions} = 1000 - 200 - 500 = 300 \]

\[ \therefore I = \text{interest amount} = 1560 - 1000 - 300 = 260 \]

\[ \text{Exp} = \text{Exposure} = A + \sum C_t (1 - t) \]

\[ = 1000 + 1000(\frac{8}{12}) - 200(\frac{6}{12}) - 500(\frac{4}{12}) = 1400 \]

\[ \therefore i_{PW} = \frac{I}{\text{Exp}} = \frac{260}{1400} = 18.57\% \]
$MacD = \frac{10 (a_{81.08}) + 800 \cdot U_{81.08}}{10 a_{81.08} + 100 \cdot U_{81.08}} = 5.989$
$i_k = k$-year spot rate = $S_k$

\[
\frac{S_5}{S_4} = \frac{f_{4, 5}}{1} \quad \text{Year 5}^{5}
\]

\[
(1 + S_5)^5 = (1 + S_4)^4 \left(1 + f_{4, 5} \right)
\]

\[
\Rightarrow f_{4, 5} = \frac{(1.075)^5}{(1.082)^4} - 1 \approx 0.47
\]
The company buys one of the 1-year bond and two of the 2-year bonds.

Prices:

\[ P_1 = \text{price of the 1-year bond} = 1000 V_{0.1} \]

\[ P_2 = \text{2-year bond} = 1000 V_{0.12} \]

\[ \therefore \text{total cost} = P_1 + P_2 = 1000 V_{0.1} + 2000 V_{0.12} = 250.3 \]
\[ PV = \frac{5000}{1 + c} + \frac{5000}{(1 + j)(1 + c)} + \frac{5000}{(1 + k)(1 + j)(1 + c)} \]

\[ = \frac{5000}{1.0651} + \frac{5000}{(1.0726)(1.0651)} + \frac{5000}{(1.0726)(1.0726)(1.0651)} \]

\[ = 13151.85 \quad \text{(round-off error)} \]

Note: We could get an exact expression for PV by recognizing:

\[ 1 + c = \frac{(1 + s_2)^2}{1 + s_1}, \quad (1 + j)(1 + c) = \frac{(1 + s_3)^3}{1 + s_1}, \quad \text{and} \]

\[ (1 + k)(1 + j)(1 + c) = \frac{(1 + s_4)^4}{1 + s_1}. \]

Then, by (1), we get

\[ PV = \frac{5000(1 + s_1)}{(1 + s_2)^2} + \frac{5000(1 + s_1)}{(1 + s_3)^3} + \frac{5000(1 + s_1)}{(1 + s_4)^2} \approx 13153 \]
21 Nov, 2005

Statements II and III are true for statement I, we achieve immunization at interest rate i, if using i,

(i) \( PV(assets) = PV(liabilities) \)

(ii) \( \text{Duration}(assets) = \text{Duration}(liabilities) \) can be MacD or Mod D

(iii) \( \text{Convexity}(assets) > \text{Convexity}(liabilities) \)

Statement I is false.
PV = 5000(1.07) + 5000(1.07)^2 + ... + 5000(1.07)^{20}

= \frac{5000(1.07)}{1.05} \left(1 + \frac{1.07}{1.05} + ... + \frac{1}{1.05}\right)

= \frac{5000(1.07)}{1.05} \times \frac{1.05^{20} - 1}{1.05 - 1} = 122,633.60

PV = P

P = 60 \cdot \frac{1}{1.07} + 60 \cdot \frac{1}{1.08} + 1060 \cdot \frac{1}{1.09} = \frac{60}{1.07} + \frac{60}{1.08^2} + \frac{1060}{(1.09)^3}

\therefore P \approx 926.03
From #33, we have \( P = 926.03 \)

\[
926.03 = 60a_\overline{31}\overline{i} + 1000\overline{v}_i^3
\]

\[
\Rightarrow i \overline{v}^m = 8.92\% 
\]

We're given \( P(0.08) = 100 \)

and \( P'(0.08) = -700 \)

By definition, \( \text{Mod } D = -\frac{P'(i)}{P(i)} \)

so \( i = 0.08 \), \( \text{Mod } D = -\frac{-700}{100} = 7 \)

Also, \( \text{Mod } D = 2 \cdot \text{Mac } D \Rightarrow \text{Mac } D = \text{Mod } D \cdot (1 + i) \)

\( \Rightarrow \text{Mac } D = 7(1.08) = 7.56 \)

\[
\text{Mac } D = \frac{D^n + 2D^{n+1} + 3D^{n+2} + \cdots}{D^n + D^{n+1} + D^{n+2} + \cdots} = \frac{D \cdot (\text{Ia})_{\overline{n}}} {D \cdot a_{\overline{n}}} 
\]

\( (\text{Ia})_{\overline{n}} = \frac{\hat{a}_{\overline{n}} - n\overline{v}^n}{i} \overset{n=\infty}{\Rightarrow} (\text{Ia})_{\overline{\infty}} = \frac{\hat{a}_{\overline{\infty}} - 0}{i} = \frac{1}{i} - 0 = \frac{1}{i} \cdot \overline{d} \)

\( a_{\overline{\infty}} = \frac{1}{i} \Rightarrow \text{Mac } D = \frac{\overline{v}_d}{i} = \frac{1}{\overline{d}} = \frac{1}{i} (1 + i) = 1 \overline{\frac{i}{i}} \)
\[
M_{ACD} = \frac{Dv + 2(1.02D)\nu^3 + 3(1.02^2D)\nu^3 + \ldots}{Dv + 1.02D\nu^3 + 1.02^2D\nu^3 + \ldots}
= \frac{D(\nu + 2(1.02)\nu^3 + 3(1.02^2)\nu^3 + \ldots)}{D(\nu + 1.02\nu^3 + 1.02^2\nu^3 + \ldots)}
\]

Let \(n\) = numerator and \(d\) = denominator.

The expression defining \(n\) is "georithmic". So use the following trick:

\[
n = \nu + 2(1.02)\nu^3 + 3(1.02^2)\nu^3 + \ldots
- \left[1.02\nu \cdot n = (1.02)\nu^3 + 2(1.02^2)\nu^3 + \ldots\right]
\]

\[
n - 1.02\nu \cdot n = \nu + 1.02\nu^3 + 1.02^2\nu^3 + \ldots \quad \text{(now it's geometric)}
\]

\[
n(1 - 1.02\nu) = \frac{\nu}{1 - 1.02\nu}
\]

\[
\therefore n = \frac{\nu}{(1 - 1.02\nu)^2}
\]

\[
d = \nu + 1.02\nu^3 + 1.02^2\nu^3 + \ldots \quad \text{Geometric} \quad \frac{\nu}{r = 1.02\nu} = \frac{\nu}{1 - 1.02\nu}
\]

\[
\therefore M_{ACD} = \frac{n}{d} = \frac{\nu(1 - 1.02\nu)^2}{\nu(1 - 1.02\nu)} = \frac{1}{1 - 1.02\nu} \cdot \frac{\nu = 1.05}{3.5}
\]
Let \( m \) = \# of Bond I needed
\[ n = \# \text{ of Bond II needed} \]

For every Bond I bought, there will be a coupon of 40 and the redemption value of 1000, both paid at the end of six months. Since we have \( m \) of them, there will be a payment of \( 1040m \) paid at the end of six months, from the purchase of the \( m \) Bond I's.

Likewise, from the purchase of \( n \) Bond II's, there will be a payment of \( 25n \) at the end of six months and \( 1025n \) at the end of one year.

The timeline is:

\[ A: \]
\[
\begin{align*}
&1040m & 1025n \\
\text{semi-annual periods} & 0 & 1 & 2
\end{align*}
\]

Exact matching \( \Rightarrow \) \[
\begin{align*}
1040m + 25n &= 1000 \\
1025n &= 1000
\end{align*}
\]

\[
\therefore n = \frac{1000}{1025} \approx 0.97561 \Rightarrow 1040m + 25(\frac{1000}{1025}) = 1000
\]

\( \Rightarrow m = 0.93809 \). Buy .93809 Bond I's & .97561 Bond II's
The payment from Mortgage I is \( X(1.06) \) at \( t=1 \).
The payment from Mortgage II is \( R \) at \( t=1 \) \( \& t=2 \),
where \( Y = R \cdot a_{\overline{3}|0.01} \implies R = \frac{Y}{a_{\overline{3}|0.01}} \).

We have

\[ L:\]

\[ yrs\quad 0\quad 1\quad 2 \]

\[ 2000\quad 1000 \]

\[ A:\] \[
\begin{array}{ll}
X(1.06) & \quad \frac{Y}{a_{\overline{3}|0.01}} \\
\end{array}
\]

\[
\therefore \quad \left\{ \begin{array}{c}
X(1.06) + \frac{Y}{a_{\overline{3}|0.01}} = 2000 \\
\frac{Y}{a_{\overline{3}|0.01}} = 1000 \\
\end{array} \right.
\]

\[ \implies Y = 1808.02 \quad \therefore \quad X = 943.40 \]

\[ \therefore \quad X + Y = 2751.42 \]
Let \( m = \# \text{ of Bond I's needed} \) and \( n = \# \text{ of Bond II's} \).

Each Bond I costs \( P^I = 1000 \, V_{0.06} \).

Each Bond II costs \( P^II = 1000 \, V_{0.07}^2 \).

The timeline is:

\[
\begin{array}{c}
0 & 1 & 2 & 3 \\
L & \frac{1000}{0.06} & \frac{1000}{0.07} \rightarrow \frac{2000}{1.065} \\
A & 1000m & 1000n & 1000n (1.065) \\
\end{array}
\]

\[
\begin{cases}
1000m = 1000 \\
1000n (1.065) = 2000
\end{cases} \Rightarrow m = 1, \quad n = \frac{2}{1.065}
\]

\[ \therefore \text{ total purchase price is } P = P^I \cdot m + P^II \cdot n \]

\[ P = 1000 \, V_{0.06} (1) + 1000 \, V_{0.07}^2 \left( \frac{2}{1.065} \right) = 2583.66 \]
We'll do this one by changing the face amounts of the bonds, which is mathematically equivalent to buying different numbers of standard face amount bonds.

Let $F_5$ denote the face amount of the 5-year bond and $F_{10}$ the 10-year bond.

$$PV(L) = 35000\,a_{15} \quad PV(A) = F_5 \cdot v^5 + F_{10} \cdot v^{10}$$

We only need to focus on the numerators of the MacD's. For the liabilities, we have ($N_L =$ numerator for liabilities)

$$N_L = 35000 \,(Ia)_{15}$$

For the assets

$$N_A = 5\,F_5 \cdot v^5 + 10\cdot F_{10} \cdot v^{10}$$

we solve the system

$$\begin{cases} 
35000\,a_{15} = F_5 \cdot v^5 + F_{10} \cdot v^{10} \\
35000\,(Ia)_{15} = 5\,F_5 \cdot v^5 + 10\,F_{10} \cdot v^{10}
\end{cases}$$

$$\therefore 35000\,a_{15} - 35000\,(Ia)_{15} = 5\,F_5 \cdot v^5 \quad i = .062$$

$$\Rightarrow F_5 \neq \text{(I was about to solve for } F_5, \text{ but after reading the question, the answer is } F_5 \cdot v^5.)$$

The amount invested in 5-year bonds is $F_5 \cdot v^5 = \frac{35000\,a_{15} - 35000\,(Ia)_{15}}{5}$

$$\therefore F_5 \cdot v^5 = 208556.21$$
Since the bond is bought at par, $P = 1000$.

Remark: They don't tell you this, but this $P=1000$ is based on an aear equal to 7.2%, the interest rate that we're "changing from".

\[ \Delta P = -P \cdot \text{modD} \cdot \Delta i \]

The interest rate changes from .072 to .08

\[ \therefore \Delta i = 0.08 - 0.072 = 0.008 \]

$P$ is the price at the old interest rate of .072

\[ \therefore P = 1000 \]

\[ \text{modD} \text{ is the modD at the old interest rate of .072} \]

\[ \therefore \text{modD} = \frac{7.959}{1.072} \]

\[ \therefore \Delta P = -1000 \left( \frac{7.959}{1.072} \right) (0.008) \approx 59.40 \]

\[ \Rightarrow P^{\text{new}} = P^{\text{old}} + \Delta P = 1000 - 59.40 = 940.60 \]
\[ f = f_{[2,3]} = \text{third year, one-year forward rate} \]

The prices given are prices of zero-coupon bonds with redemption value = 1, at the given maturity times. I.e.

\[ v_{S_1} = .9542 \implies 1 + s_1 = \frac{1}{.9542} \]

\[ v_{S_2} = .90703 \implies (1 + s_2)^2 = \frac{1}{.90703} \]

\[ v_{S_3} = .85892 \implies (1 + s_3)^3 = \frac{1}{.85892} \]

We seek \( f = f_{[2,3]} \):

\[ f = \frac{v_{S_3}}{(1 + s_2)^2} \]

\[ \implies f = \frac{(1 + s_3)^3}{(1 + s_2)^2} = \frac{.90703}{.85892} = .056 \]
\[ P = 5000 \]
\[ F = C = 5000 \]
\[ r = .05 \]

\[ 5000 = 250 \frac{a}{g_1} + 5000u^8 \implies i = \frac{5000}{5000} \]

Note: Generally, if \( P = F = C \), then \( i = r \)

\[ d_1, d_2 \]

Changing the valuation date resets the time values

\[ d_1 = \frac{250(1)u + 250(2)u^2 + \cdots + 250(\pi)u^\pi + 5000(\pi)u^\pi}{250 \frac{a}{g_1} + 5000 \pi} \]

\[ = \frac{250(Ia)q_1 + 35000 \pi}{250 \frac{a}{g_1} + 5000 \pi} \approx \frac{30378.46}{5250} \approx 5.78637 \]

\[ d_2 = \frac{250(1)u + 250(2)u^2 + \cdots + 250(\pi)u^\pi + 5000(\pi)u^\pi}{250 a_{\pi 1} + 5000 \pi} \]

\[ = \frac{250(Ia)q_1 + 35000 \pi}{250 a_{\pi 1} + 5000 \pi} = \frac{30378.46}{5000} \]

\[ \therefore \frac{d_1}{d_2} = \frac{5000}{5250} = .95238 \]
Let \( m \) = \# of Bond A needed
\( n \)
\( p \) = Bond C

Then the timeline is:

\[
\begin{align*}
L : & \quad 0 \quad 1 \quad 2 \quad 3 \\
& \quad 99 \quad 102 \quad 100
\end{align*}
\]

\[
\begin{align*}
A : & \quad 107m \quad 100n \quad 105p \\
& \quad 5p \quad 5p 
\end{align*}
\]

\[
\begin{align*}
107m + 5p &= 99 \\
100n + 5p &= 102 \\
105p &= 100
\end{align*}
\]

We seek \( m \).

\[
p = \frac{100}{105}
\]

\[
107m + 5\left(\frac{100}{105}\right) = 99
\]

\[
\therefore m = 0.8807
\]
Since the liability has an asset before and after, we get full immunization by setting $\text{PV}(A) = \text{PV}(L)$ and, simultaneously, $\text{Mac}D^A = \text{Mac}D^L$. Since $\text{PV}(A) = \text{PV}(L)$ we can just set the numerator of the $\text{Mac}D^A_n$ equal to the numerator of the $\text{Mac}D^L_n$. We get

\[
\begin{align*}
6000 \cdot 2^4 &= A \cdot 2^2 + B \cdot 2^6 \\
6000(4) \cdot 2^4 &= 2A2^2 + 6B2^6
\end{align*}
\]

We can eliminate $B$ by multiplying the first equation by 6 and subtracting the second equation, getting

\[
12000 \cdot 2^4 = 4A2^2 \implies A = 3000 \cdot 2^{0.05}
\]

Then

\[
6000 \cdot 2^4 = (3000 \cdot 2^3) \cdot 2^2 + B \cdot 2^6
\]

\[
\implies B = 3000 \cdot (1.05)^2
\]

\[
\therefore |A - B| = 586.41
\]
\[
\begin{align*}
L: \quad 12000 & \quad \text{A:} \quad 5000 \quad \text{B} \\
\rho \cdot \eta^L &= \rho \cdot \eta^A \quad \text{see notation from previous problem} \\
12000 \cdot \eta^8 &= 5000 \cdot \eta^5 + B \cdot \eta^{8+b} \\
12000 \cdot (8) \cdot \eta^8 &= 5000 \cdot (5) \cdot \eta^5 + B \cdot (8+b) \cdot \eta^8 \\
\text{Substitution is easier to use in this problem. From the first equation} & \quad B \cdot \eta^{8+b} = 12000 \cdot \eta^8 - 5000 \cdot \eta^5 \\
\text{Substituting into the second equation yields} & \quad 96000 \cdot \eta^8 = 25000 \cdot \eta^5 + (12000 \cdot \eta^8 - 5000 \cdot \eta^5) \cdot (8+b) \\
\Rightarrow \quad b &= \frac{96000 \cdot \eta^8 - 25000 \cdot \eta^5}{12000 \cdot \eta^8 - 5000 \cdot \eta^5} - 8 = 2.50765 \\
\text{Then, from} & \quad B \cdot \eta^{8+b} = 12000 \cdot \eta^8 - 5000 \cdot \eta^5 \\
\Rightarrow \quad B &= 7039.27 \\
\therefore \quad \frac{B}{b} &= 2807
\end{align*}
\]
\[
\begin{align*}
0.04 & \quad n^L = n^A \\
(95000 v^5 = A v^3 + B v^9) \quad (9) \\
95000(5) v^5 = 2A v^3 + 9B v^9 \\
\end{align*}
\]

\[
380000 v^5 = 7A v^3 \Rightarrow A = \frac{380000 v^3}{7}
\]

Then \[
95000 v^5 = \frac{380000 v^3}{7} \cdot v^2 + B v^9
\]

\[
\Rightarrow B = \frac{285000 (1.04)^7}{7}
\]

\[
\therefore \frac{A}{B} = \frac{380000}{285000 (1.04)^7} = 1.0132
\]

Remark: If you don't want to get exact values, then just approximate \( A \) & \( B \) to get \( \frac{A}{B} \).
\[ B = 5000(1.09) + 2600(1.09)^{1/2} = 8164.48 \]

\[ i = i_{TW} \implies 1 + i = \frac{5200}{5000} \cdot \frac{B}{7800} \implies i = 0.0886 \]

\[ i = i_{TW} \implies 1 + i = \frac{120000}{100000} \cdot \frac{130000}{150000} \cdot \frac{100000}{80000} \implies i = 30\% \]

AV (deposits) = 750 \( \bar{S}_{18}, 0.07 \)

Tuition costs for the 18th school year = \( T_{18} = 6000(1.05)^{17} \)

The excess at \( t=17 \) is \( E = 750 \bar{S}_{18}, 0.07 - 6000(1.05)^{17} \)

\( T_{19} = 6000(1.05)^{18} \). The excess accumulates to \( E(1.07) \)

\[ X = 6000(1.05)^{18} - E(1.07) = 1870.25 \]
We seek \( f = f_{4,5} \)
\[
(1+0.045)^4 \cdot (1+f) = (1+0.05)^5 \implies f = \frac{\left(\frac{1.05}{1.045}\right)^5 - 1}{4} \approx 0.115
\]

The \( X \) at \( t=0 \) accumulates to \( X \cdot (1+0.05)^{0.4} \). However, since each bond is zero coupon, \( P = C \cdot V^K \), or \( \frac{P}{C} = V^K \), which is what's given in the table. E.g. the 94% in the table means \( 94 = \text{sdf using the 6-month spot rate. Then the } \text{saf} = (1+0.05)^{0.4} = \text{sdf}^{-1} = (0.94)^{-1} \).

The \( X \) at \( t=0 \) accumulates to \( X \cdot (0.94)^{-1} \).

We should technically use forward rates to accumulate the \( X \)'s at other times. However, since "bond prices will not change during the 6-month period", we can use the given spot rate information. E.g. the \( X \) at time 3 accumulates to \( X \cdot \text{saf} = X \cdot (\text{gdf})^{-1} = X \cdot (0.97)^{-1} \). Using this logic, we have

\[
100000 = X \left[ (0.94)^{-1} + (0.95)^{-1} + (0.96)^{-1} + (0.97)^{-1} + (0.98)^{-1} + (0.99)^{-1} \right]
\]
\[
\implies X \approx 16078.29
\]
\( (1 + S_3)^3 = (1.04)(1.06)(1.08) \implies S_3 = 1.059874 \)

\[
\begin{align*}
1 + i_{TW} = & \frac{52000}{50000} \cdot \frac{62000}{60000} \cdot \frac{55000}{52000} \\
\implies i_{TW} = & .136 (= i_{DW})
\end{align*}
\]

\[
i_{DW} = .136 \implies 55000 = 50000(1+.136) + 8000(1+.136)^{\frac{2}{3}} - 10000(1+.136(1-t))
\]

\[
\implies t = .5886
\]

\[
\text{Annuity A:}
\]

\[
\text{MacD}^A = \frac{(1)(1) + (1.2) + (1.2)^2}{1 + v + v^2} = \frac{v + 2v^2}{1 + v + v^2} = .93
\]

\[
\implies v + 2v^2 = .93 + .93v + .93v^2 \implies 1.07v^2 + .07v - .93 = 0
\]

\* by quadratic formula, \( v \approx 0.90015 \)

\[
\text{Annuity B:}
\]

\[
\text{MacD}^B = \frac{(1)(1) + (1) + (1)^2 + (2)^3}{1 + v + v^2 + v^3} = \frac{v + 2v^2 + 3v^3}{1 + v + v^2 + v^3} = \frac{v + 2v^2 + 3v^3}{1 + v + v^2 + v^3} = \frac{99015}{1369}
\]
\[
\text{Mac D} = \frac{40000(2) V^3 + 25000(3) V^3 + 100000(4) V^3}{40000 V^2 + 25000 V^3 + 100000 V^4} = 3.314
\]
\[
\text{Mac D} = 30 = \frac{\frac{1}{c} + 1(1) V + 1(3) V^2 + 1(3) V^3 + \cdots}{1 + V + V^2 + V^3 + \cdots} = \frac{V + 2V^2 + 3V^3 + \cdots}{1 + \alpha^{\infty}} = \alpha^{\infty}
\]

Remarks:
1) The numerator is \((\frac{P}{c} + \frac{Q}{c^3})\) with \(P = Q = 1\)

I.e. the numerator equals \(\frac{1}{c} + \frac{1}{c^3} = \frac{1 + c}{c^3}\)

2) The denominator equals \(1 + \frac{1}{c} = \frac{1 + c}{c}\) \(\quad \text{Same}\)

(If you're thinking \(\alpha^{\infty}\), then it's \(\alpha^{\infty} = \frac{1}{1} = \frac{1}{\gamma(1c)} = \frac{1 + c}{c}\))

\(\therefore\) \(\text{Mac D} = 30 = \frac{(\frac{1 + c}{c^3})}{(\frac{1 + c}{c})} = \frac{1}{c} \Rightarrow c = \frac{1}{30}\)

\(\text{Mod D} = V \cdot \text{Mac D} = \frac{1}{1 + \frac{1}{30}} (30) = 29.03\)
\[
\begin{align*}
\text{Exam FM} & \quad \text{Sample Questions} \\
\text{act} &= 1.10 \\
A: X &= Y \\
\left\{ \begin{align*}
PVL &= PV(A) \\
\max B^b = \max A^a \\
n^c &= n^a
\end{align*} \right. & \quad \Rightarrow \quad & \\
500 \cdot 2 + 1000 \cdot 2^4 &= X + Y \cdot 2^3 \\
500 \cdot 2 + 4(100) \cdot 2^4 &= 0(x) + 3 \cdot 2^3 \\
\therefore Y &= \frac{500(1.1)^2 + 4000 \cdot 2}{3} = 1413.79
\end{align*}
\]

and \( X = 75.36 \) (Either A or B is correct.)

The easiest way to proceed is to change the interest rate a small amount, to say 12% (±2% is good), and see how this changes the \( PVL \) vs \( PV(A) \). Be sure to use exact values for \( X \) & \( Y \), and not rounded values. At \( i = 12 \) we get

\[
\begin{align*}
PVL &= 500 \cdot 2 + 1000 \cdot 2^4 = 1081.94665 \\
PV(A) &= X + Y \cdot 2_{12}^3 = 1081.665438
\end{align*}
\]

Since \( PVL > PV(A) \), even though by not much, immunization has not be achieved.
\[ l = 0.04 \]

\[
\begin{align*}
\left\{ \begin{array}{c}
PV(L) = PV(A) \\
\sum_l^n = \sum_A^n
\end{array} \right. \\
1000000 \cdot 2^8 = 300000 \cdot 2^6 + X \cdot 2^\gamma \\
8(1000000) \cdot 2^8 = 6(300000) \cdot 2^6 + \gamma \cdot X \cdot 2^\gamma
\end{align*}
\]

Using substitution, \( X \cdot 2^\gamma = 1000000 \cdot 2^8 - 300000 \cdot 2^6 \)

\[
\gamma = \frac{8000000 \cdot 2^8 - 1800000 \cdot 2^6}{1000000 \cdot 2^8 - 300000 \cdot 2^6} = 8.96068
\]

Then, \( X = (1000000 \cdot 2^8 - 300000 \cdot 2^6) \cdot (1.04)^\gamma \approx 701,458.26 \)
Exam FM Sample Questions

\[ PV(L) = PV(A) \]
\[ \text{Mac } D^L = \text{Mac } D^A \]
\[ i = 0.07 \]

Since \( PV(L) = 573v^2 + 701v^5 = 1000 \), and noting that \( PV(A) = \frac{\text{Price}}{\text{Total}} \), we can eliminate answer choices C and D since in both cases \( PV(A) = 1274 \).

\[ \text{Mac } D^L = \frac{2(573)v^2 + 5(701)v^5}{573v^2 + 701v^5} = 3.5 \]

Note that 3.5 is halfway between 1 and 6. For choice B, \( \text{Mac } D_{\text{portfolio}} = \frac{572}{1000} (1) + \frac{428}{1000} (6) \).

We could do the arithmetic, or notice the weight on 1 is greater than the weight on 6, and so \( \text{Mac } D \) is closer to 1 than to 6. I.e. \( \text{Mac } D < 3.5 \) (Eliminate B).

Answer choice A is correct. Notably, I think they mean you have invested in an

Since answer choice A has an asset before the first liability, and after the last one, if \( \text{Mac } D^A = 3.5 \), then it is our answer. The redemption values are:

- Bond A: 500(1.07)
- Bond B: 500(1.07)^6

\[ \text{Mac } D = \frac{535v + 500(1.07)^6}{535v + 500(1.07)^6} v^6 = \frac{3500}{1000} = 3.5, \quad \text{Answer} = A \]
\[
\begin{align*}
&\text{Sample Questions} \\
&i = 10\% \\
&\begin{cases}
PV(L) = PV(A) \\
&A_i^n = A^n
\end{cases} \Rightarrow \begin{align*}
402.11 \cdot a_{31.10} &= X \cdot i + Y \cdot i^3 \\
402.11 \cdot (A_{31.10}) &= X \cdot i + 3Y \cdot i^3
\end{align*}
\end{align*}
\]

Subtracting first equation from the second yields:

\[
402.11 (A_{31.10} - a_{31.10}) = 2Y i^3
\]

\[
\Rightarrow Y \approx 623.27
\]

\[
P \cdot X = (402.11 a_{31} - Y i^3)(1.1) \approx 584.88
\]

\[
\therefore \text{1-year bond costs } X \cdot i \approx 531.72
\]

\[
\text{and 3-year bond costs } Y \cdot i^3 \approx 468.27
\]
The easiest way to proceed is as follows:

1) \( PV(A) = PV(L) = 9.697 \) (This does not eliminate any answer choices, as you can check.)

2) \( \text{Mac} D^A = \text{Mac} D^L = 15.24 \)

   Note: For a k-year zero coupon bond, \( \text{Mac} D = k \)
   Check answer choices for \( \text{Mac} D^A \)

   (A) \( \text{Mac} D = \frac{3077}{9697} (5) + \frac{6620}{9697} (20) = 15.24 \)

   (B) \( \text{Mac} D = \ldots \) (or notice the weight on 5 is \( \frac{6620}{9697} \), larger than 0.5. So the weighted average of 5 & 20 is closer to 5, and so cannot equal 15.24.) Eliminate (B)

   (C) \( \text{Mac} D \) is closer to 20 than 15, so \( \text{Mac} D \neq 15.24 \)
   Eliminate (C)

   (D) Same as (C). Eliminate D.

   (E) \( \text{Mac} D = \frac{9232}{9697} (15) + \frac{465}{9697} (20) = 15.24 \)

   Answer is either (A) or (E)

3) \( \text{Mac} C^A > \text{Mac} C^L \)

   Check answer choices (A) & (E)

   (A) \( \text{Mac} C^A = \frac{3077}{9697} (5)^2 + \frac{6620}{9697} (20)^2 = 28.1 \) \( > \) \( \text{Mac} C^L \)

   \( \therefore \) Answer = (A)

   Remark: For (E), \( \text{Mac} C^A = \frac{9232}{9697} (5)^2 + \frac{465}{9697} (20)^2 = 23.3 \) \( < \) \( \text{Mac} C^L \)
Let $X =$ face amount (redemption value) for Bond $H$

\[
Y = \quad I
\]

\[
Z = \quad J
\]

Note: $P^H =$ Price of Bond $H = X \times 0.11$

$P^E = \quad I = Y \times 0.11$

$P^J = \quad J \text{ "sell at par" } Z$

You can see this also as follows

$P^J = 0.12Z \times 1.1 + Z \times 0.11 = Z(0.12 \times 1.1 + 0.11) = Z$

The timeline is:

\[
\begin{array}{c|c|c}
& 0 & 1 \\
\hline
A: & X & Y \\
\hline
B: & 0.12Z & 1.12Z \\
\end{array}
\]

\[
\begin{align*}
11000 &= X + 0.12Z \\
12100 &= Y + 1.12Z \\
\end{align*}
\]

\[\Rightarrow \text{ the price } P = P^H + P^E + P^J = \frac{X}{1.1} + Y + 0.11 + Z
\]

Highest Profit $\Rightarrow$ Smallest price for bonds.

One way to proceed is as follows: Consider $P$ for each answer choice.

(A) $P = 9091 + 8264 + 2145 = 19500$

(B) $P = 10000 + 10000 = 20000$

(C) $P = 19821$

(D) $P = 19641$

(E) $P = 19625$

\[\begin{align*}
X &= 9703 \text{ at } t=1, \text{ provides } X + 1.12Z = 10000 \\
Y &= 8264 \text{ at } t=1, \text{ provides } Y + 0.11Z = 10182 \\
Z &= 2145 \text{ at } t=2, \text{ provides } 1.12Z = 12100
\end{align*}\]
Let \( X \) = face amount of bond in part i)
\( Y = \) ii)
\( Z = \) iii)

The timeline is:

\[ \begin{array}{c|c|c}
   & 25000 & 20000 \\
0 & & \\
1 & 1.0675 & \\
2 & 1.0454 & \\
\end{array} \]

\[ \begin{align*}
25000 &= 1.0675X + 0.045Y \\
20000 &= 1.045Y + Z
\end{align*} \]

Remarks: 1) There are infinitely many solutions to this system.
2) We seek the solution that gives the smallest discounted value of assets, which is obtained by discounting at the largest interest rates.
3) For parts i) & ii), the "at par" means the bonds are bought at par. So \( F = C = P \), and so \( r = i \). I.e. the bond in i) yields \( i = 0.0675 \) and the bond in ii) yields \( i = 0.045 \). The bond in iii) yields \( i = 0.05 \), as given.

From remarks 2) \& 3), do not use the bond in i). I.e. take \( Y = 0 \). We can solve for \( X \) by using \( Z \), or just recognize we'll be discounting the 25000 using \( i = 0.0675 \) and the 20000 using \( i = 0.05 \), obtaining

\[ \frac{25000}{1.0675} + \frac{20000}{(1.05)^2} = 41559.79 \]