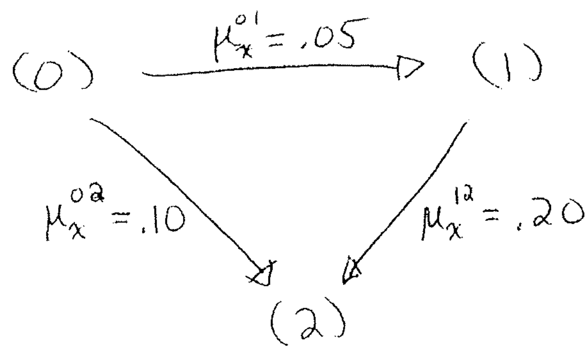
1) (See Video Solution)



(a) $\;{}_5p_x^{00} = e^{-.75}$

(b) $\;{}_5p_x^{01} = e^{-.75} - e^{-1}$

(c) $\;{}_5p_x^{02} = 1 - 2e^{-.75} + e^{-1}$

2)

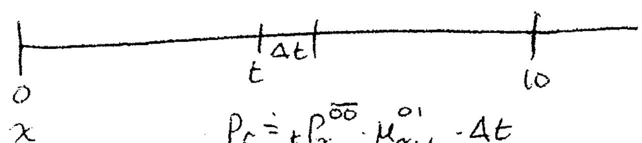
$$(0) \xrightarrow{\mu^{01}_{x+t} = .02t} (1)$$

(a) $${}_{10}p_x^{00} = {}_{10}p_x^{\overline{00}} = e^{-\int_0^{10} \mu_{x+t}^{0\tau}\,dt} = e^{-\int_0^{10} .02t\,dt}$$

$$= e^{-.01t^2\big|_0^{10}} = e^{-1}$$

(b) $${}_{10}p_x^{01} = 1 - {}_{10}p_x^{00} = 1 - e^{-1} \quad \text{(easy way)}$$

(hard way):



$$Pr \doteq {}_tp_x^{\overline{00}} \cdot \mu_{x+t}^{01} \cdot \Delta t$$

$$\therefore \; {}_{10}p_x^{01} = \int_0^{10} {}_tp_x^{\overline{00}}\, \mu_{x+t}^{01}\,dt \qquad {}_np_x^{\overline{00}} = e^{-\int_0^n .02t\,dt} = e^{-.01n^2}$$

$$\Rightarrow {}_tp_x^{\overline{00}} = e^{-.01t^2}$$

$$\therefore \; {}_{10}p_x^{01} = \int_0^{10} e^{-.01t^2}(.02t)\,dt$$

$u = .01t^2$

$du = .02t\,dt$

t	u
0	0
10	1

$$= \int_0^1 e^{-u}\,du = e^{-u}\Big|_1^0 = 1 - e^{-1} \quad \text{(as above)}$$

4)

$$\begin{tikzcd}
(0) \arrow[r, shift left, "\mu^{01}=.1"] \arrow[d, "\mu^{03}=.1"'] & (1) \arrow[l, shift left, "\mu^{10}=.2"] \arrow[d, "\mu^{12}=.2"] \arrow[dl, "\mu^{13}=.2"'] \\
(3) & (2) \arrow[l, "\mu^{23}=.1"]
\end{tikzcd}$$

(a) ${}_0p_x^{01} = 0$ since given (x) is in state 0 at time 0 it's impossible to be in state 1 at time 0

(b) ${}_5p_x^{\overline{11}} = e^{-\int_0^5 \mu_{x+t}^{1\bullet}\,dt}$ $\qquad \mu_{x+t}^{1\bullet} = \mu^{10} + \mu^{12} + \mu^{13} = .2+.2+.2 = .6$

$$\Longrightarrow {}_5p_x^{\overline{11}} = e^{-.6(5)} = e^{-3}$$

(c) ${}_{10}p_x^{22} = {}_{10}p_x^{\overline{22}}$ $\qquad \mu_{x+t}^{2\bullet} = \mu^{23} = .1$

$$\Longrightarrow {}_{10}p_x^{22} = {}_{10}p_x^{\overline{22}} = e^{-.1(10)} = e^{-1}$$

(d) $${}_t\dot{p}_x^{23} = \overbrace{\underbrace{{}_tp_x^{20}}_{=0} \cdot \mu_{x+t}^{03} + \underbrace{{}_tp_x^{21}}_{=0} \cdot \mu_{x+t}^{13} + {}_tp_x^{22} \cdot \mu_{x+t}^{23}}^{=\text{rate in}} - \overbrace{{}_tp_x^{23} \cdot \underbrace{\mu_{x+t}^{3\bullet}}_{=0}}^{=\text{rate out}}$$

$$\therefore {}_t\dot{p}_x^{23} = {}_tp_x^{22} \cdot \mu_{x+t}^{23} \qquad \mu_{x+t}^{23} = .1 \qquad {}_tp_x^{22} \underset{\text{part (c)}}{\overset{\text{see}}{=}} e^{-.1t}$$

$$\Longrightarrow {}_t\dot{p}_x^{23} = .1e^{-.1t}$$

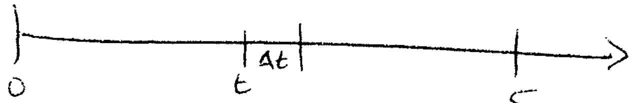
(e) $${}_t\dot{p}_x^{10} = \overbrace{{}_tp_x^{11} \cdot \underbrace{\mu_{x+t}^{10}}_{=.2} + {}_tp_x^{12} \cdot \underbrace{\mu_{x+t}^{20}}_{=0} + {}_tp_x^{13} \cdot \underbrace{\mu_{x+t}^{30}}_{=0}}^{=\text{rate in}} - \overbrace{{}_tp_x^{10} \cdot \underbrace{\mu_{x+t}^{0\bullet}}_{=.1+.1=.2}}^{=\text{rate out}}$$

$$\therefore {}_t\dot{p}_x^{10} = .2\,{}_tp_x^{11} - .2\,{}_tp_x^{10}$$

5)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
(0) \arrow[r, "\mu^{01}_{xy} = .01"] \arrow[d, "\mu^{02}_{xy} = .02"'] & (1) \arrow[d, "\mu^{13}_{y} = .02"] \\
(2) \arrow[r, "\mu^{23}_{x} = .01"] & (3)
\end{tikzcd}
$$

We seek ${}_5p^{01}$:



$Pr = {}_tp^{00} \cdot \mu^{01} \cdot \Delta t \cdot {}_{(5-t)}p^{11}$

$$\therefore \; {}_5p^{01} = \int_0^5 {}_tp^{00} \cdot \mu^{01} \cdot {}_{(5-t)}p^{11} \, dt$$

$${}_np^{00} = {}_np^{\overline{00}} = e^{-\int_0^n \mu^{0\bullet} dt} = e^{-\int_0^n (.01 + .02) dt} = e^{-.03n}$$

$${}_np^{11} = {}_np^{\overline{11}} = e^{-\int_0^n \mu^{1\bullet} dt} = e^{-.02n}$$

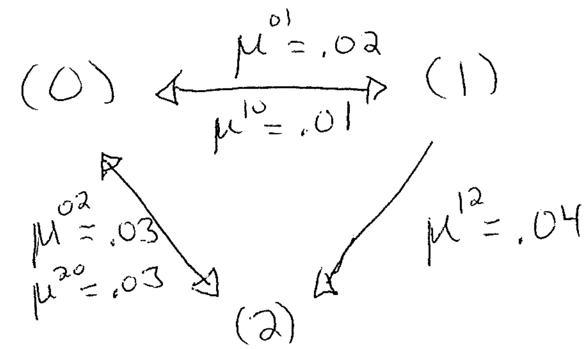
$$\therefore \; {}_5p^{01} = \int_0^5 e^{-.03t} (.01) \, e^{-.02(5-t)} \, dt$$

$$= .01 e^{-.1} \int_0^5 e^{-.01t} \, dt$$

$$= e^{-.1} \cdot e^{-.01t} \Big|_5^0 = e^{-.1}(1 - e^{-.05})$$

$$= e^{-.1} - e^{-.15}$$

6)



(a) ${}_t\dot{p}_x^{00} = \left[\text{rate in}\right] - \left(\text{rate out}\right)$

$$\text{rate in} = {}_tp_x^{01} \cdot \mu^{10} + {}_tp_x^{02} \cdot \mu^{20}$$

$$\text{rate out} = {}_tp_x^{00} \cdot \left(\mu^{01} + \mu^{02}\right)$$

$$\therefore \; {}_{.5}\dot{p}_x^{00} = \left[{}_{.5}p_x^{01} \cdot \mu^{10} + {}_{.5}p_x^{02} \cdot \mu^{20}\right] - \left({}_{.5}p_x^{00} \cdot (\mu^{01} + \mu^{02})\right)$$

$$= \left[.01(.01) + .015(.03)\right] - \left(.975(.02 + .03)\right) = -0.0482$$

(b) EM: $y(t+h) = y(t) + h \cdot \dot{y}(t)$

$$y(t) = {}_tp_x^{00}$$

$$\left.\begin{matrix} t = 0.5 \\ h = 0.1 \end{matrix}\right\} \Rightarrow \; {}_{.6}p_x^{00} = {}_{.5}p_x^{00} + .1 \cdot {}_{.5}\dot{p}_x^{00}$$

$$= 0.975 + .1(-0.0482)$$

$$= 0.97018$$

7)

$$\begin{tikzcd}
(0) \arrow[r, "\mu_x^{01}=.04"] \arrow[d, "\mu_x^{02}=.02"'] & (1) \arrow[d, leftrightarrow, "\mu_x^{13}=.001e^{0.1x}=\mu_x^{31}"] \\
(2) \arrow[ur, "\mu_x^{21}=.01"] \arrow[r, "\mu_x^{23}=.03"'] & (3)
\end{tikzcd}$$

(a) ${}_{10}P_{30}^{12} = 0$ since it is not possible for someone in state 1 to ever enter state 2.

(b) If someone in state 0 ever leaves state 0, then they can never return to state 0,

$\therefore \; {}_{10}P_{30}^{00} = {}_{10}P_{30}^{\overline{00}} = e^{-\int_0^{10} \mu_{30+t}^{0\tau}\,dt} = e^{-\int_{30}^{40} \mu_x^{0\tau}\,dx}$

$\mu_{30+t}^{0\tau} = .04 + .02 = .06 = \mu_x^{0\tau}$

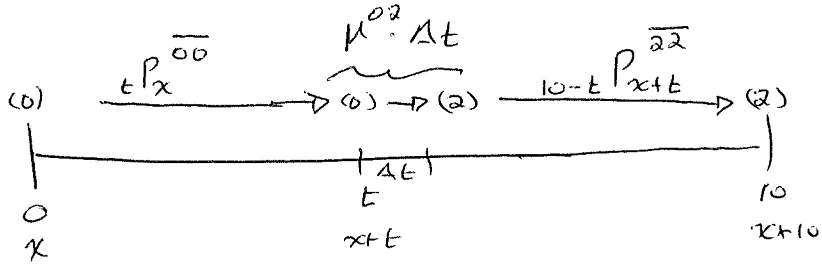
$\therefore \; {}_{10}P_{30}^{00} = {}_{10}P_{30}^{\overline{00}} = e^{-\int_0^{10} .06\,dt} = e^{-\int_{30}^{40} .06\,dx} = e^{-.06(10)} = e^{-.6}$

More generally,

${}_{n}P_{30}^{00} = {}_{n}P_{30}^{\overline{00}} = e^{-.06(n)}$

7) (Continued)

(C) ${}_{10}P_x^{02}$:



$$P_r = {}_tP_x^{\overline{00}} \cdot \mu^{02} \cdot {}_{10-t}P_{x+t}^{\overline{22}} \cdot \Delta t$$

$\mu^{01}+\mu^{02}=.06$ $\qquad$ $\mu^{21}+\mu^{23}=.04$

$$ {}_{10}P_x^{02} = \int_0^{10} e^{-.06t} \cdot (.02) \cdot e^{-.04(10-t)}\, dt$$

$$= .02 \cdot e^{-.4} \int_0^{10} e^{-.02t}\, dt$$

$$= e^{-.4} \cdot e^{-.02t} \Big|_{10}^{0} = e^{-.4}\left(1 - e^{-.2}\right)$$

More generally,

$$ {}_nP_x^{02} = \int_0^{n} e^{-.06t} \cdot (.02) \cdot e^{-.04(n-t)}\, dt$$

$$= .02\, e^{-.04n} \cdot \int_0^{n} e^{-.02t}\, dt$$

$$= e^{-.04n} \cdot e^{-.02t} \Big|_{n}^{0}$$

$$= e^{-.04n} \cdot \left(1 - e^{-.02n}\right)$$

7) (Continued)

(d)

$$\dot{p} = \text{"rate in"} - \text{"rate out"}$$

$$\begin{tikzcd}
(0) \arrow[r] \arrow[d] & (1) \arrow[d, leftrightarrow] \\
(2) \arrow[ur] \arrow[r] & (3)
\end{tikzcd}$$

$${}_t\dot{P}^{01}_{30} = \left[{}_tP^{00}_{30} \cdot \mu^{01}_{30+t} + {}_tP^{02}_{30} \cdot \mu^{21}_{30+t} + {}_tP^{03}_{30} \cdot \mu^{31}_{30+t}\right] - \left[{}_tP^{01}_{30} \cdot \mu^{13}_{30+t}\right]$$

$${}_t\dot{P}^{03}_{30} = \left[{}_tP^{01}_{30} \cdot \mu^{13}_{30+t} + {}_tP^{02}_{30} \cdot \mu^{23}_{30+t}\right] - \left[{}_tP^{03}_{30} \cdot \mu^{31}_{30+t}\right]$$

With $t=10$, we have:

$${}_{10}P^{00}_{30} = e^{-.6} \quad \text{(See part (b))}$$

$${}_{10}P^{01}_{30} \approx .2587 \quad \text{(Given)}$$

$${}_{10}P^{02}_{30} = e^{-.4}(1-e^{-.2}) \quad \text{(See part (c))}$$

$${}_{10}P^{03}_{30} \approx .071 \quad \text{(Given)}$$

$$\mu^{01}_{30+10} = \mu^{01}_{40} = .04$$

$$\mu^{21}_{30+10} = \mu^{21}_{40} = .01$$

$$\mu^{31}_{30+10} = \mu^{31}_{40} = .001\,e^{.1(40)} = .001e^{4} = \mu^{13}_{40} = \mu^{13}_{30+10}$$

$$\mu^{23}_{30+10} = \mu^{23}_{40} = .03$$

$\therefore$

$${}_{10}\dot{P}^{01}_{30} \approx \left[e^{-.6}(.04) + e^{-.4}(1-e^{-.2})(.01) + .071(.001e^{4})\right] - \left[.2587(.001e^{4})\right] \approx .01293$$

and

$${}_{10}\dot{P}^{03}_{30} \approx \left[.2587(.001e^{4}) + e^{-.4}(1-e^{-.2})(.03)\right] - \left[.071(.001e^{4})\right] \approx .01389$$

7) (Continued)

(e) EM: $y(t+h) \approx y(t) + h \cdot \dot{y}(t)$

With $y(t) = {}_tP_x$, we get

$${}_{t+h}P_x \approx {}_tP_x + h \cdot {}_t\dot{P}_x \quad \left(\begin{matrix}\text{any } i \\ \text{any } j\end{matrix}\right)$$

With $t=10$ and $h=.2$, we get $(x=30)$

$${}_{10.2}P_x^{01} \approx {}_{10}P_x^{01} + .2\,{}_{10}\dot{P}_x^{01} \approx .2587 + .2(.01293) \approx .2613$$

$${}_{10.2}P_x^{03} \approx {}_{10}P_x^{03} + .2\,{}_{10}\dot{P}_x^{03} \approx .071 + .2(.01389) \approx .0738$$

<u>Remark</u>: ${}_{10.2}P_{30}^{00} = e^{-.06(10.2)} \approx .5423$ (See part (b))

and ${}_{10.2}P_{30}^{02} = e^{-.04(10.2)}(1 - e^{-.02(10.2)}) \approx .1227$ (See part (c))

Then ${}_{10.2}P_{30}^{00} + {}_{10.2}P_{30}^{01} + {}_{10.2}P_{30}^{02} + {}_{10.2}P_{30}^{03}$

$\approx .5423 + .2613 + .1227 + .0738 = 1.0001$ (should be equal to 1; the extra .0001 is round-off error)

(f)

$${}_{10.4}P_{30}^{01} \approx {}_{10.2}P_{30}^{01} + .2\,{}_{10.2}\dot{P}_{30}^{01}$$

$${}_{10.2}P_{30}^{01} \approx .2613 \text{ (see part (e))}$$

For ${}_{10.2}\dot{P}_{30}^{01}$, go back to the KDE for ${}_t\dot{P}_{30}^{01}$ in part d.

We get

$${}_{10.2}\dot{P}_{30}^{01} = .5423(.04) + .1227(.01) + .0738(.001e^{4.02}) - .2613(.001e^{4.02}) \approx .01248$$

$$\therefore \; {}_{10.4}P_{30}^{01} \approx .2613 + .2(.01248) \approx .2638$$

7) (Continued)

(f) (Continued)

Likewise $\quad {}_{10.4}p_{30}^{03} \approx {}_{10.2}p_{30}^{03} + .2 \; {}_{10.2}\dot{p}_{30}^{03}$

$${}_{10.2}p_{30}^{03} \approx .0738 \quad \text{(see part (e))}$$

For ${}_{10.2}\dot{p}_{30}^{03}$, go back to the KDE for ${}_{t}\dot{p}_{30}^{03}$ in part d

We get

$${}_{10.2}\dot{p}_{30}^{03} \approx .2613(.001e^{4.02}) + .1227(.03) - .0738(.001e^{4.02}) \approx .01412$$

$$\therefore \; {}_{10.4}p_{30}^{03} \approx .0738 + .2(.01412) \approx .0766$$

<u>Remark</u>: We've found in part (f) that

$${}_{10.4}p_{30}^{01} \approx .2638 \quad \text{and} \quad {}_{10.4}p_{30}^{03} \approx .0766$$

We also have $\quad {}_{10.4}p_{30}^{00} = e^{-.06(10.4)} \approx .5358$

and $\quad {}_{10.4}p_{30}^{02} = e^{-.04(10.4)}\left(1 - e^{-.02(10.4)}\right) \approx .1239$

Then $\quad {}_{10.4}p_{30}^{00} + {}_{10.4}p_{30}^{01} + {}_{10.4}p_{30}^{02} + {}_{10.4}p_{30}^{03}$

$\approx .5358 + .2638 + .1239 + .0766 = 1.0001$ (round-off error again)