Solutions to L-TAM Module 1 Section 3 Exercises

1. (a) We seek ${}_{20}p_{25}$, the probability that a 25-year old survives 20 years, to age 45. This probability is the proportion of all 25 year olds who live to be 45. Therefore, using life table notation and the ILT, ${}_{20}p_{25} = \frac{l_{45}}{l_{25}} \overset{ILT}{=} \frac{99{,}033.9}{99{,}871.1}$.

   (b) We seek ${}_{12}q_{33}$, the proportion of all 33-year olds who die within 12 years, by age 45. Therefore, ${}_{12}q_{33} = \frac{{}_{12}d_{33}}{l_{33}} = \frac{l_{33}-l_{45}}{l_{33}} \overset{ILT}{=} \frac{99{,}629.3-99{,}033.9}{99{,}629.3} = \frac{595.4}{99{,}629.3}$

   (c) We seek ${}_{2|}q_{27}$, the proportion of all 27-year olds who defer death for 2 years and die in the following 1 year period. I.e. we seek the proportion of all 27-year olds who die between ages 29 and 30. Therefore, ${}_{2|}q_{27} = \frac{d_{29}}{l_{27}} = \frac{l_{29}-l_{30}}{l_{27}} \overset{ILT}{=} \frac{99{,}757.7-99{,}727.3}{99{,}815.9}$, and so ${}_{2|}q_{27} = \frac{30.4}{99{,}815.9}$

   (d) We seek ${}_{4|3}q_{38}$, the proportion of all 38-year olds who die between ages 42 and 45. Therefore, ${}_{4|3}q_{38} = \frac{{}_{3}d_{42}}{l_{38}} = \frac{l_{42}-l_{45}}{l_{38}} \overset{ILT}{=} \frac{99{,}229.8-99{,}033.9}{99{,}433.3} = \frac{195.9}{99{,}433.3}$

2. Since the l_x function is linear, then there is a (global) uniform distribution of deaths between ages 0 and 100. I.e., this is a DML(100) mortality model. Therefore,

$$f_x(t) = \frac{1}{100-x}$$
$${}_tq_x = t \cdot q_x = \frac{t}{100-x}$$
$${}_tp_x = 1 - \frac{t}{100-x} = \frac{100-x-t}{100-x}$$
$\mu_x = \frac{1}{100-x}$, or equivalently, $\mu_{x+t} = \frac{1}{100-x-t}$

$\overset{o}{e}_x = \frac{100-x}{2}$ and $e_x = \frac{100-x-1}{2} = \overset{o}{e}_x - \frac{1}{2}$

You should be able to easily adapt these formulas to a general terminal age, ω, instead of using 100 above.

(a) We can use either ${}_{22}p_{50} = \frac{l_{72}}{l_{50}}$ or ${}_{22}p_{50} \overset{DML}{=} \frac{100-50-22}{100-50}$. Either way, ${}_{22}p_{50} = 0.56$.

(b) $\overset{o}{e}_0 \overset{DML}{=} \frac{100-0}{2} = 50$

(c) $\overset{o}{e}_{27} \overset{DML}{=} \frac{100-27}{2} = 36.5$

(d) $e_{42} \overset{DML}{=} \frac{100-42-1}{2} = 28.5$

3. You should recognize this mortality model as GDML(100,2). Therefore,

$${}_np_x = \left(\frac{100-x-n}{100-x}\right)^2$$

$$\mu_x(t) = \frac{2}{100-x-t}$$

$$\overset{o}{e}_x = \frac{100-x}{2+1}$$

Instead of using 100 and 2 above, you should be able to adapt these formulas to a general terminal age, ω, and general value, α, for GDML(ω, α) mortality.

(a) We can use either ${}_{50}p_{25} = \frac{l_{75}}{l_{25}}$ or ${}_{50}p_{25} \overset{GDML(100,2)}{=} \left(\frac{100-25-50}{100-25}\right)^2$. Either way, ${}_{50}p_{25} = \frac{1}{9}$.

(b) $\mu_{47}(18) \overset{GDML(100,2)}{=} \frac{2}{100-47-18} = \frac{2}{35}$

(c) $\overset{o}{e}_{55} \overset{GDML(100,2)}{=} \frac{100-55}{3} = 15$

4. Compare to the previous problem. This is not GDML mortality. There's nothing we can do here but to use basic facts.

(a) ${}_2p_5 = \frac{l_7}{l_5} = \frac{500(100-49)}{500(100-25)} = \frac{51}{75}$

(b) Notice that the terminal age is 10. Therefore

$${}_tp_7 = \frac{l_{7+t}}{l_7} = \begin{cases} \dfrac{100-(7+t)^2}{51} & \text{if } t \le 3 \\ 0 & \text{if } t > 3 \end{cases}$$

Then $\overset{o}{e}_7 = \int_0^3 \frac{100-(7+t)^2}{51}\,dt = \int_0^3 \frac{51-14t-t^2}{51}\,dt = \frac{81}{51}$

5. (a) Since there is a uniform distribution of deaths, there will be the same number of deaths over any 10 year period. Therefore, since ${}_{10}d_x = 500$ then ${}_{10}d_{x+5} = 500$. I'll use the shorthand ${}_nd$ to represent the number of deaths over an n-year period when given a uniform distribution of deaths. So here, ${}_{10}d = 500$. Note that a uniform distribution of deaths also implies that ${}_nd \overset{UDD}{=} n \cdot d$. I'm using the "omitted 1" notation; namely, ${}_1d = d$ represents the number of deaths over a 1-year period.

   Technically, we should restrict ages as to not extend beyond the terminal age, but it's not likely you'll be "tricked" with this on exam questions. E.g. for terminal age $\omega = 100$, if we were given ${}_{10}d_{95} = 500$, then we would actually have ${}_5d_{95} = 500$ since no one survives past age 100. Again, this is not likely to show up on exams, and I'll not mention this restriction in future exercises and solutions.

   (b) Since there is a uniform distribution of deaths and since there are 500 deaths over any 10 year period, then there will be 50 deaths over any one year period. Therefore, over any three year period there will be 150 deaths, and so ${}_3d_x = 150$. We can capture this information symbolically by saying

$$ {}_{10}d_x \overset{UDD}{=} 10d_x = 500 \Rightarrow d_x \overset{UDD}{=} d = 50 \Rightarrow {}_3d_x \overset{UDD}{=} {}_3d \overset{UDD}{=} 3d = 150. $$

   You should recognize how to extend this to a general situation.

6. (See Video Solution)

   (a) ${}_{10}q_x = .2$

   (b) ${}_5p_x = .9$

   (c) ${}_{5|10}q_x = .2$

   (d) ${}_{10}q_{x+5} = \frac{10(.02)}{1-5(.02)} = \frac{2}{9}$

7. (a) Since there's a uniform distribution of deaths, then ${}_{10}d_{50.25} \overset{UDD}{=} {}_{10}d = 500$. Then $l_{30} = 3500$, $l_{40} = 3000$, etc. Since $l_{20} = 4000$ and $4000 = 500(8)$, there are 8 10-year periods from age 20 to the terminal age, ω. Therefore, $\omega = 20 + 80 = 100$. Note that we could have broken it down on a per year basis by recognizing that the number of deaths per year is $d = 50$. Then, since $4000 = 50(80)$, there are 80 years from age 20 to the terminal age. Again we get $\omega = 20 + 80 = 100$.

   (b) A uniform distribution of deaths implies we have a DML mortality model. Since we saw from part (a) that the terminal age is 100, then ${}_{35}p_{40} \overset{DML}{=} \frac{100-40-35}{100-40} = \frac{25}{60}$.

8. (a) For the 25 years between age 0 and age 25, there are 500 deaths. Since deaths are uniformly distributed, there are $d = \frac{500}{25} = 20$ deaths per year during this period. Therefore between age 0 and age 15, there are 20(15) = 300 deaths, and so $l_{15} = l_0 - 300 = 1000 - 300 = 700$.

   (b) For the 50 years between age 25 and age 75, there are 100 deaths. Since deaths are uniformly distributed, there are $d = \frac{100}{50} = 2$ deaths per year during this period. Therefore between age 25 and age 35, there are 2(10) = 20 deaths, and so $l_{35} = l_{25} - 20 = 500 - 20 = 480$.

   (c) First, note that there is a uniform distribution of deaths of 20 per year for the first 25 years and a uniform distribution of deaths of 2 per year for the following 50 years. Because the number of deaths per year changes, this is not a DML mortality model. Instead of ${}_np_x \overset{DML}{=} \frac{\omega - x - n}{\omega - x}$, we use ${}_np_x = \frac{l_{x+n}}{l_x}$. Using parts (a) and (b) we get ${}_{20}p_{15} = \frac{l_{35}}{l_{15}} = \frac{480}{700}$

9. Since mortality is DML(90), then the age at death random variable, $X = T_0$ has a uniform distribution over the interval [0,90]. Therefore, T_{30}, the random variable representing the time until death of a 30-year old, is uniformly distributed over the interval [0,60]. Parts (a) and (b) follow from uniform distribution facts.

   (a) $E[T_{30}] = \frac{0+60}{2} = 30$

   (b) $Var(T_{30}) = \frac{(60-0)^2}{12} = 300$

10. Constant force mortality (with force of mortality $\mu_x = \mu$) is equivalent to T_x, the random variable representing the time until death of an *x*-year old, following an exponential distribution with mean equal to $\frac{1}{\mu}$. Parts (a) and (b) follow from exponential distribution facts.

   (a) $E[T_x] = \frac{1}{\mu} = \frac{1}{.025} = 40$

   (b) $Var(T_x) = \left(\frac{1}{\mu}\right)^2 = 40^2 = 1600$

11. Recognize the mortality model as CF(.02).

(a) As we saw in the previous section, for a constant force of mortality, ${}_tp_x$ will not depend on x. (This is the memory-less property of the exponential distribution.) In fact, we saw that ${}_tp_x \overset{CF}{=} p^t$ where $p = e^{-\mu}$. Therefore, in this case, with $\mu = .02$, we have ${}_tp_x = e^{-.02t}$.

Note that we could have used life table notation to arrive at the same answer. I.e., ${}_tp_x = \frac{l_{x+t}}{l_x} = \frac{500e^{-.02(x+t)}}{500e^{-.02x}} = e^{-.02t}$.

(b) As in the previous problem, we recognize that a $\mu = .02$ constant force model is equivalent to T_x following an exponential distribution with mean $\frac{1}{\mu} = \frac{1}{.02} = 50$. Since $\overset{o}{e}_x = E[T_x]$, the mean of T_x, then $\overset{o}{e}_x = 50$.

Although recognizing the equivalence between constant force mortality models and exponential distributions will save you time, we could arrive at the same answer using $\overset{o}{e}_x = \int_0^\infty {}_tp_x\, dt = \int_0^\infty e^{-.02t}\, dt$, or even $\overset{o}{e}_x = \int_0^\infty t \cdot f_x(t)\, dt = \int_0^\infty .02te^{-.02t}\, dt$. In the last integral we use the fact that $f_x(t) = -\, {}_t\dot{p}_x = .02e^{-.02t}$.

(c) It will generally be very easy in a constant force mortality model to determine $e_x = E[K_x]$ by using $e_x = \sum_{k=1}^{\infty} {}_kp_x$, since this series will be geometric. Here we get, $e_x = p_x + {}_2p_x + {}_3p_x + \cdots = p + p^2 + p^3 + \cdots = \frac{p}{1-p}$ where $p = e^{-\mu} = e^{-.02}$. Therefore $e_x = \frac{e^{-.02}}{1-e^{-.02}}$.

What follows is an alternative way to determine e_x that has to do with a relationship between the (continuous) exponential distribution and the (discrete) geometric distribution. Note that $e_x = E[K_x]$ where $K_x = \lfloor T_x \rfloor$ is the curtate future lifetime random variable for (x). We know that T_x has an exponential distribution with mean $\frac{1}{\mu} = 50$. A useful fact is that if T_x has an exponential distribution, then the discrete random variable $\lfloor T_x \rfloor$ has a geometric distribution.

There are a couple of subtleties that arise with using this fact, though. First, there are two common ways to describe a geometric distribution; namely, 1) the number of failures until the first success, and 2) the number of trials until the first success. Described as the number of failures until the first success, the support for the random variable is 0, 1, 2, ... and the mean of the random variable is $\frac{\text{Pr (failure)}}{\text{Pr (success)}}$. Described as the number of trials until the first success, the support for the random variable is 1, 2, 3, ... and the mean of the random variable is $\frac{1}{\text{Pr (success)}}$. Since the support of $K_x = \lfloor T_x \rfloor$ is 0, 1, 2, ... then we should be using the "number of failures until the first success" description, and we get $e_x = E[K_x] = \frac{\text{Pr (failure)}}{\text{Pr (success)}}$.

The second subtlety has to do with how we define a success. Let's think about this by considering what $K_x = k$ means. For example, $K_x = 2$ means that (x) died between ages x + 2 and x + 3. So we can think of $K_x = 2$ as having two years without death before the first year with death. Going back to "success-failure" terminology, $K_x = 2$ is the event of having two failures before the first success. Therefore, we should define success as a year with death and failure as a year without. Then using p-q actuarial notation $\Pr(\text{success}) = q \overset{CF}{=} 1 - e^{-\mu t}$, $\Pr(\text{failure}) = p \overset{CF}{=} e^{-\mu t}$, and $e_x = \frac{e^{-.02}}{1-e^{-.02}}$ as before.

12. Since the function l_x is exponential, this is another constant force problem. We can determine the force of mortality by solving $e^{-\mu} = 0.95$, and so $\mu = -\ln(0.95)$. Now we can easily proceed as in the last problem. We get

(a) ${}_tp_x = e^{-\mu t} = e^{-(-\ln(0.95))t} = e^{\ln(0.95)^t} = (0.95)^t$

(b) $\overset{o}{e}_x = \frac{1}{\mu} = \frac{-1}{\ln(0.95)}$

For (c), we again get $e_x = p + p^2 + p^3 + \cdots = \frac{p}{1-p}$, where this time $p = 0.95$. Therefore, $e_x = \frac{0.95}{0.05} = 19$.

Alternative derivations for (a) and (b) are:

(a) ${}_tp_x = \frac{l_{x+t}}{l_x} = \frac{500(0.95)^{(x+t)}}{500(0.95)^x} = (0.95)^t$

(b) $\overset{o}{e}_x = \int_0^\infty {}_tp_x\,dt = \int_0^\infty (0.95)^t\,dt = \frac{1}{\ln(0.95)}(0.95)^t|_0^\infty = \frac{-1}{\ln(0.95)}$

13. (See Video Solution)

(a) $p_x = 0.9$

${}_2p_x = 0.81$

${}_3p_x = 0.729$

${}_kp_x = (0.9)^k$

(b) $e_x = 9$