Solutions to MLC Module 1 Section 4 Exercises

1. (a) Since there is a uniform distribution of deaths between neighboring ages, and since there are 40 deaths between ages 50 and 51, then there must be 10 deaths for any 0.25 year period between ages 50 and 51. Therefore, there are 10 deaths between ages 50 and 50.25, and so $l_{50.25} = 600 - 10 = 590$.

With a UDD assumption, instead of going through this process each time, from now on we will use linear interpolation between integer ages. That is, we’ll determine the life table value at fractional ages by taking the weighted arithmetic mean of the life table values at corresponding neighboring ages. In this problem, we get

$$l_{50.25}^{UDD} = .75 \cdot l_{50} + .25 \cdot l_{51} = .75 \cdot 600 + .25 \cdot 560 = 590 \text{ as before.}$$

Notice that since the age is 50.25, then the weights are 0.25 and 0.75. (Weights always sum to 1.) Since 50.25 is closer to age 50, the higher weight of 0.75 gets assigned to $l_{50}$. This sort of logic will apply in both UDD and CF problems.

(b) When using a CF assumption, we determine the life table value at fractional ages by taking the weighted geometric mean of the life table values at corresponding neighboring ages. Assigning the weights as is part (a), we get

$$l_{50.6}^{CF} = (l_{50})^4 \cdot (l_{51})^6 = (600)^4 \cdot (560)^6 \approx 575.66963$$

2. From the given information, we have $l_{47} = 1000$, $l_{48} = l_{47} \cdot p_{47} = 1000(.95) = 950$, and $l_{49} = l_{47} \cdot 2p_{47} = 1000(.90) = 900$. Therefore,

(a) $l_{48.5}^{UDD} = .5 \cdot l_{48} + .5 \cdot l_{49} = .5 \cdot (950 + 900) = 925$

(b) $l_{48.3}^{CF} = (l_{48})^7 \cdot (l_{49})^3 = (950)^7 \cdot (900)^3 \approx 934.71514$

Note, again, how the weights are assigned.

3. (a) We have $0.2q_{65.3} = 1 - 0.2p_{65.3} = 1 - \frac{l_{65.5}}{l_{65.3}}$.

$$l_{65.5}^{UDD} = .5 \cdot l_{65} + .5 \cdot l_{66}^{ULT} = .5 \cdot (7,533,964 + 7,373,338) = 7,453,651,$$

Therefore, $0.2q_{65.3} = 1 - \frac{7,453,651_{453,651}}{7,485,776.2} \approx 0.00429$
(b) \( 1.5P_{72.2} = \frac{l_{73.7}}{l_{72.2}} \)

\[
l_{73.7}^{\text{CF}} = (l_{73})^3 \cdot (l_{74})^{-7} = (5,920,394)^3 \cdot (5,664,051)^{-7} \approx 5,739,766.056, \text{ and}
\]

\[
l_{72.2}^{\text{CF}} = (l_{72})^8 \cdot (l_{73})^{-2} = (6,164,663)^8 \cdot (5,920,394)^{-2} \approx 6,115,015.95
\]

Therefore \( 1.5P_{72.2} = \frac{l_{73.7}}{l_{72.2}} \approx 0.93863 \)

4. For fractional age problems, if you’re not given \( l_x \) values, then we can develop them by starting with an arbitrary value (like 1000) at the earliest age. Looking ahead at all our calculations, we notice that the earliest age for which we’ll need an \( l_x \) value is for part (c), where we’ll need an \( l_{30} \). Likewise the latest age for which we’ll need an \( l_x \) value is for parts (b) and (d), where we’ll need an \( l_{34} \). So let’s start with \( l_{30} = 1000 \) and develop the other \( l_x \) values up to and including \( l_{34} \). We’re given \( q_{30} = 0.1, q_{31} = 0.15, q_{32} = 0.2, \) and \( q_{33} = 0.25 \). Then,

\[
l_{30} = 1000
\]

\[
l_{31} = l_{30} \cdot p_{30} = 1000(0.9) = 900
\]

\[
l_{32} = l_{31} \cdot p_{31} = 900(0.85) = 765
\]

\[
l_{33} = l_{32} \cdot p_{32} = 765(0.8) = 612
\]

\[
l_{34} = l_{33} \cdot p_{33} = 612(0.75) = 459
\]

(a) \( 0.2q_{31} = 1 - 0.2p_{31} = 1 - \frac{l_{31.2}}{l_{31}} \)

Since \( l_{31.2}^{\text{UDD}} = 0.8 \cdot l_{31} + 0.2 \cdot l_{32} = 0.8 \cdot (900) + 0.2(765) = 873, \) then

\[
0.2q_{31}^{\text{UDD}} = 1 - \frac{873}{900} = 0.03
\]

To illustrate that it won’t matter what the starting \( l_x \) value is, if we would have started with \( l_{31} = 1000 \), we would get \( l_{32} = 1000 \cdot p_{31} = 1000 \cdot (0.85) = 850 \). Then \( l_{31.2}^{\text{UDD}} = 0.8 \cdot l_{31} + 0.2 \cdot l_{32} = 0.8 \cdot (1000) + 0.2(850) = 970 \Rightarrow 0.2q_{31} = 1 - \frac{970}{1000} = 0.03 \) as above. The key idea is that for these probabilities, we’re taking ratios of the \( l_x \) values, and so the result will not depend on the starting value. For the rest of this problem, we’ll use the \( l_x \) values we computed before starting part (a).

(b) We need \( l_{33.5}^{\text{CF}} = (l_{33})^5 \cdot (l_{34})^{-5} = \frac{l_{33}}{l_{34}} = \sqrt{612} \cdot (459) \approx 530.00755. \) Then

\[
1.5P_{32}^{\text{CF}} = \frac{l_{33.5}}{l_{32}} \approx 0.69282
\]

(c) \( 0.4|0.6q_{30.8} = \frac{l_{31.2} - l_{31.8}}{l_{30.8}} \) From part (a), \( l_{31.2}^{\text{UDD}} = 873. \)

Also, \( l_{31.8}^{\text{UDD}} = 0.2(900) + 0.8(765) = 792 \) and \( l_{30.8}^{\text{UDD}} = 0.2(1000) + 0.8(900) = 920. \)

Therefore \( 0.4|0.6q_{30.8} = \frac{l_{31.2} - l_{31.8}}{l_{30.8}} = \frac{81}{920} = \frac{81}{920} \)
(d) \( 1.7|0.8 q_{31} = \frac{l_{32.7} - l_{33.5}}{l_{31}} \) from part (b), \( l_{33.5} \approx 530.00755 \).

Also, \( l_{32.7} \approx (765)^{3} \cdot (612)^{7} \approx 654.37158 \) and so \( 1.7|0.8 q_{31} \approx 1.3818 \)

5. Other than how mortality is given, this problem is very similar to the previous problem. Once we get the \( l_x \) values, we proceed as above. Looking ahead, notice in parts (a) and (c) that we’ll need \( l_{50} \) and that’s the youngest age for which we’ll need an \( l_x \) value. Instead of 1000, let’s start with an arbitrary \( l_{50} = 100 \), to change it up a little. We’re given \( q_{50} = 0.01 \), \( 1|q_{50} = 0.02 \), \( 2|q_{50} = 0.03 \), and \( 3|q_{50} = 0.04 \). Then,

\[
\begin{align*}
d_{50} &= l_{50} \cdot q_{50} = 100(0.01) = 1 \\
d_{51} &= l_{50} \cdot 1|q_{50} = 100(0.02) = 2 \\
d_{52} &= l_{50} \cdot 2|q_{50} = 100(0.03) = 3 \\
d_{53} &= l_{50} \cdot 3|q_{50} = 100(0.04) = 4
\end{align*}
\]

Therefore,

\[
\begin{align*}
l_{50} &= 100 \\
l_{51} &= l_{50} - d_{50} = 100 - 1 = 99 \\
l_{52} &= l_{51} - d_{51} = 99 - 2 = 97 \\
l_{53} &= l_{52} - d_{52} = 97 - 3 = 94 \\
l_{54} &= l_{53} - d_{53} = 94 - 4 = 90
\end{align*}
\]

(a) \( 0.2 q_{50} = 1 - 0.2 p_{50} = 1 - \frac{l_{50.2}}{l_{50}} \)

Since \( l_{50.2} = .8 \cdot l_{50} + .2 \cdot l_{51} = .8 \cdot (100) + .2(99) = 99.8 \), then

\[
0.2 q_{50} \text{ UDD} = 1 - \frac{99.8}{100} = .002
\]

(b) We need \( l_{52.5} \text{ UDD} \) \( \frac{CF}{l_{53}} \).

\[
\begin{align*}
l_{52}'^{\text{UDD}} &= (l_{52})^5 \cdot (l_{53})^5 = \sqrt{l_{52} \cdot l_{53}} = \sqrt{(97)(94)} \approx 95.48822. \text{ Then} \\
1.5 p_{51} &= \frac{l_{52.5}}{l_{51}} \approx .96453
\end{align*}
\]

(c) \( 0.6|0.4 q_{50.8} = \frac{l_{51.4} - l_{51.8}}{l_{50.8}} \)

We get \( l_{51.4} \text{ UDD} \) \( .6(99) + .4(97) = 98.2 \) and \( l_{51.8} \text{ UDD} \) \( .2(99) + .8(97) = 97.4 \) and

\[
\begin{align*}
l_{50.8} &\text{ UDD} = .2(100) + .8(99) = 99.2. \text{ Therefore} \ 0.6|0.4 q_{50.8} \text{ UDD} \frac{98.2 - 97.4}{99.2} = \frac{8}{992}
\end{align*}
\]

(d) \( 1.8|0.7 q_{51} = \frac{l_{52.9} - l_{53.5}}{l_{51}} \)

We get \( l_{52.9} \text{ UDD} \) \( (97)^2 \cdot (94)^{8} \approx 94.59248 \) and \( l_{53.5} \text{ UDD} \) \( \sqrt{(94)(90)} \approx 91.97826 \) and so

\[
1.8|0.7 q_{51} \approx .02641
\]
6. (See Video Solution)
   (a) \( t_1^{\text{UDD}} = 1 - .1t \)

   (b) \( t_p \) for \( 0 \leq t \leq 1 \) using the UDD assumption

7. (See Video Solution)
   \( 30p_{35} = \frac{1}{6} \)

8. (See Video Solution)
   \( 30p_{35} = (.72)^{15} \approx .00724 \)