Example:

\[ (0) \rightarrow (1) \quad n_{X}^{2_{0}} = 0 \]
\[ (2) \rightarrow (3) \quad n_{X}^{2_{2}} = 1 \]

\[ n_{X}^{i_{0}} = e^{-S_{X}^{i_{0}} dt} \]
where \[ S_{X}^{i_{0}} = S_{X}^{i_{1}} + S_{X}^{i_{2}} + S_{X}^{i_{3}} \]

Remark: \[ o_{X}^{i_{j}} = \begin{cases} 1 & \text{if } j = i \text{ (Initial Condition)} \\ 0 & \text{if } j \neq i \end{cases} \]

We'll use "Kolmogorov's Forward Equations" to approximate values like \( n_{X}^{i_{0}} \).

Euler's Method (EM) for approximating the solution to an initial value problem (IVP) uses:

\[ \gamma(t+h) \approx \gamma(t) + h \cdot \dot{\gamma}(t) \quad h > 0 \quad \text{(Forward equation)} \]

\[ \gamma(t) = t^{i_{j}} \quad \Rightarrow \quad \dot{\gamma}(t) = t^{i_{i}j} \quad \text{(derivative w.r.t. } t) \]

The differential equations \( t^{i_{i}j} \) are called Kolmogorov's Differential Equation (KDE's).

\[ t^{i_{j}} = "rate\ in" - "rate\ out" \], where

"rate in" = rate of transitions into state \( j \) at time \( t \)
for an \( x \)-year old who is in state \( i \) (at time 0)

"rate out" = rate of transitions out of state \( j \) at time \( t \)
for an \( x \)-year old who is in state \( i \)
Consider the above example:

1) \( t_x^0 = \{ "rate\ in\" - \{ "rate\ out\" \}
\]
\[
= [t_x^0 \cdot \mu_{x+t}^0 + t_x^0 \cdot \mu_{x+t}^{30}] - \{ t_x^0 \cdot (\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}) \}
\]

2) \( t_x^{23} = "rate\ in\" - "rate\ out"
\]
\[
= t_x^{20} \cdot \mu_{x+t}^{03} \quad \text{and} \quad t_x^{23} \cdot \mu_{x+t}^{30} \quad \text{and} \quad t_x^{23} \cdot \mu_{x+t} \quad = 0
\]
\[
\Rightarrow t_x^{23} = 0 \Rightarrow t_x^{23} \text{ is constant by observation, } t_x^{23} = 0 \text{ for all } t
\]

3) Likewise \( t_x^{22} = 0 \Rightarrow t_x^{22} \text{ is constant by observation, } t_x^{22} = 1 \text{ for all } t
\]

3) \( t_x^{10} = "rate\ in\" - "rate\ out\"
\]
\[
= [t_x^{11} \cdot \mu_{x+t}^1 + t_x^{13} \cdot \mu_{x+t}^{30}] - \{ t_x^{10} \cdot (\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}) \}
\]

See Handout Example
The non-zero transition rates for a 4-state model are:

\[
\begin{align*}
\mu_x^{01} &= .04 & \mu_x^{02} &= .02 & \mu_x^{21} &= .01 \\
\mu_x^{23} &= .03 & \mu_x^{13} &= .001e^{0.1x} = \mu_x^{31}
\end{align*}
\]

(a) Determine \(10p_{30}^{12}\)

(b) Determine \(10p_{30}^{00}\) More generally, determine \(n p_{30}^{00}\) for any \(n \geq 0\).

(c) Determine \(10p_{30}^{02}\) More generally, determine \(n p_{30}^{02}\) for any \(n \geq 0\).

For parts (d), (e), and (f), you are also given \(10p_{30}^{01} \approx 0.2587\) and \(10p_{30}^{03} \approx 0.0710\).

(d) Determine \(10p_{30}^{01}\) and \(10p_{30}^{03}\)

(e) Use an iteration of Euler's Forward Equation with step size equal to 0.2 to approximate \(10.2p_{30}^{01}\) and \(10.2p_{30}^{03}\)

(f) Perform another iteration of Euler's Forward Equation with step size equal to 0.2
to approximate \(10.4p_{30}^{01}\) and \(10.4p_{30}^{03}\)
(d) \(10 P_{30}^{0.01}\): First do these calculations using \( t \), then plug in the number (here \( t = 10 \))

\[
\dot{P}_{30}^{0.01} = \text{"rate in" - "rate out"}
\]

\[
= tP_{30}^{0.00} M_{30+t}^{0.01} + tP_{x}^{0.02} M_{x+t}^{0.21} + tP_{x}^{0.03} M_{x+t}^{0.31} - tP_{x}^{0.01} M_{x+t}^{0.13}
\]

\[
= 10 P_{30}^{0.01} = e^{-b} (0.04) + (e^{-0.4} - e^{-0.6})(0.01) + (0.0710)(0.001e^{0.40})
\]

\[
- (0.2587)(0.001e^{0.40}) =
\]

\(10 P_{30}^{0.03} \) (homework) to 4 digits