Solutions to MLC Module 1 Section 1 Exercises

1. Generally, $p + q = 1$, with the same decorations on p and q. So ${}_{20}p_{50} = 1 - {}_{20}q_{50} = 1 - .75 = .25$.

2. Note that we usually omit the duration value when it equals 1. I.e. $q_{30} = {}_1q_{30}$ and $p_{30} = {}_1p_{30}$. As with the above problem, $q_{30} = 1 - p_{30} = 1 - .95 = .05$

3. Recognize this as a problem where we need to use factorization of the p's. (There is no factorization of q's.) Start with the earliest age and shortest duration associated to this age, then proceed as prompted. We get ${}_{30}p_{20} = {}_{10}p_{20} \cdot {}_{20}p_{30} \Rightarrow {}_{30}p_{20} = (.9)(.6) = .54$

4. Proceed as in the previous problem. We get ${}_{20}p_{40} = {}_5p_{40} \cdot {}_{15}p_{45} \Rightarrow {}_{15}p_{45} = \frac{.7}{.9} = \frac{7}{9}$

5. Recognize that we can factor the p's to get ${}_{15}p_{35}$ and then ${}_{15}q_{35} = 1 - {}_{15}p_{35}$. We get ${}_{15}p_{35} = \frac{{}_{35}p_{35}}{{}_{20}p_{50}} = \frac{.32}{.4} = .75 \Rightarrow {}_{15}q_{35} = .25$.

6. Use factorization of p's. The earliest age is 20. The durations associated to age 20 are 30 (from the symbol ${}_{30}p_{20}$) and 10 (from the symbol ${}_{10}q_{20}$). So the shortest duration associated with the earliest age of 20 is 10. Start with age 20 and duration 10 and proceed with factoring the p's using what the given information leads us to use. We get ${}_{50}p_{20} = {}_{10}p_{20} \cdot {}_{30}p_{30} \cdot {}_{10}p_{60} \Rightarrow {}_{50}p_{20} = \frac{7}{8} \cdot {}_{30}p_{30} \cdot \frac{3}{4} = \frac{21}{32} \cdot {}_{30}p_{30}$. Now using the earliest age of 20 with the other duration of 30, we get the factorization ${}_{50}p_{20} = {}_{30}p_{20} \cdot {}_{20}p_{50} \Rightarrow {}_{50}p_{20} = \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$. Therefore $\frac{3}{8} = \frac{21}{32} \cdot {}_{30}p_{30} \Rightarrow {}_{30}p_{30} = \frac{4}{7} \Rightarrow {}_{30}q_{30} = \frac{3}{7}$.

7. Since ${}_2p_x = p_x \cdot p_{x+1}$ then $.1 = e^{-\mu} \cdot e^{-3\mu} = e^{-4\mu} \Rightarrow \mu = \frac{\ln(.1)}{-4} = \frac{\ln(10)}{4}$

8. We're given ${}_tp_0 = 1 - (.01t)^2, 0 \le t \le 100$. We'll refer to 0-year olds as newborns. Note that ${}_0p_0 = 1$, as it should be, since this is just the probability that a newborn is alive at time 0 (right now). Also, note that ${}_{100}p_0 = 0$. The earliest age at which the probability of surviving to is 0 is called the terminal age, and we generally denote it by ω (omega). So in this model the terminal age is $\omega = 100$.

(a) Factoring p's gives ${}_{50}p_0 = {}_{30}p_0 \cdot {}_{20}p_{30}$. Since ${}_{30}p_0 = 1 - (.3)^2 = .91$ and ${}_{50}p_0 = 1 - (.5)^2 = .75$, then ${}_{20}p_{30} = \frac{75}{91}$

(b) Use (a) as a guide and factor p's: ${}_{30+t}p_0 = {}_{30}p_0 \cdot {}_tp_{30}$. Then ${}_{30}p_0 = .91$, and ${}_{30+t}p_0 = 1 - \left(.01(30 + t)\right)^2 = 1 - (.3 + .01t)^2 = .91 - .006t - .0001t^2$. Therefore, ${}_tp_{30} = \frac{.91 - .006t - .0001t^2}{.91}$.

Note that $0 \le t \le 70$ for this expression since the terminal age is $\omega = 100$ and this is the survival function for a 30-year old. Also, note that ${}_0p_{30} = \frac{.91-.006(0)-.0001(0)^2}{.91} = 1$, as it should be, since this is just the probability that a 30-year old is alive at time 0 (right now).

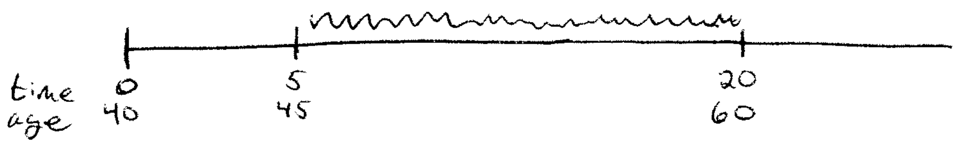
9. (See Video Solution)
   (a) ${}_2q_{30} = .28$.

   (b) ${}_2q_{34} = .8$

   (c) ${}_3q_{30} = .496$

   (d) ${}_3q_{34} = .94$

10. We seek the probability that (40) dies between ages 45 and 60.



Since ${}_{5|15}q_{40} = \begin{cases} {}_{20}q_{40} - {}_5q_{40} \\ {}_5p_{40} - {}_{20}p_{40} \\ {}_5p_{40} \cdot {}_{15}q_{45} \end{cases}$ there are several ways to proceed. Recognizing that

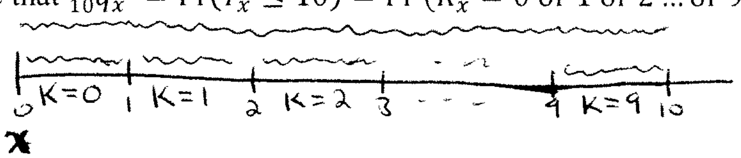
${}_5p_{40} \cdot {}_{15}p_{45} = {}_{20}p_{40}$, we get ${}_5p_{40} = \frac{.63}{.7} = .9$. Therefore

${}_{5|15}q_{40} = \begin{cases} {}_{20}q_{40} - {}_5q_{40} = .37 - .1 \\ {}_5p_{40} - {}_{20}p_{40} = .9 - .63 \\ {}_5p_{40} \cdot {}_{15}q_{45} = .9 \cdot (.3) \end{cases}$ and so ${}_{5|15}q_{40} = .27$

11. ${}_{10|20}q_{30} = {}_{10}p_{30} - {}_{30}p_{30} \Rightarrow .2 = .85 - {}_{30}p_{30} \Rightarrow {}_{30}p_{30} = .65$

12. ${}_{30|20}q_{40} = {}_{30}p_{40} \cdot {}_{20}q_{70} \Rightarrow .19 = (.2) \cdot {}_{20}q_{70} \Rightarrow {}_{20}q_{70} = \frac{19}{20} = .95$

13. Note that ${}_{10}q_x = \Pr(T_x \le 10) = \Pr(K_x = 0 \text{ or } 1 \text{ or } 2 \ldots \text{or } 9)$

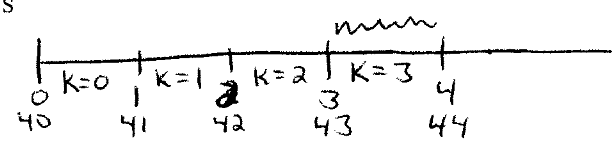


$\Pr(K_x = k) = {}_{k|}q_x$ and so ${}_{10}q_x = q_x + {}_{1|}q_x + \cdots + {}_{9|}q_x$. Note that $q_x = {}_{0|}q_x$.
Therefore ${}_{10}q_x = .01 + .02 + \cdots + .09 = .45$ and so ${}_{10}p_x = .55$

14. Since $f_{40}(t)$ is the pdf for T_{40}, we have $\int_0^{30} f_{40}(t)dt = \Pr(T_{40} \leq 30) = {}_{30}q_{40}$ and $\int_{10}^{\infty} f_{40}(t)dt = \Pr(T_{40} > 10) = {}_{10}p_{40}$. Then ${}_{10|20}q_{40} = {}_{30}q_{40} - {}_{10}q_{40} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. We could have just have easily used p's, getting ${}_{10|20}q_{40} = {}_{10}p_{40} - {}_{30}p_{40} = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}$.

15. As in the previous problem, $\int_{20}^{\infty} f_{40}(t)dt = \Pr(T_{40} > 20) = {}_{20}p_{40}$. Since we seek ${}_{20}p_{40}$ then I'll use the formula ${}_{15|20}q_{25} = {}_{15}p_{25} \cdot {}_{20}q_{40}$. Using the given information, we then get $.18 = (.9) \cdot {}_{20}q_{40} \Rightarrow {}_{20}q_{40} = .2$ and so $\int_{20}^{\infty} f_{40}(t)dt = {}_{20}p_{40} = .8$

16. Note that ${}_{k|}q_{40} = \Pr(K_{40} = k)$. Sometimes it helps to draw a timeline. Using $k = 3$ as an example, the symbol ${}_{3|}q_{40}$ represents the probability that (40) dies between ages 43 and 44. The corresponding timeline is



We see that ${}_{3|}q_{40} = \Pr(K_{40} = 3)$.

Therefore, the probability distribution table for K_{40} is

K_{40}	Pr
0	$\Pr(K_{40} = 0) = q_{40} = {}_{0|}q_{40} = \frac{1}{50}$
1	$\Pr(K_{40} = 1) = {}_{1|}q_{40} = \frac{1}{50}$
2	$\Pr(K_{40} = 2) = {}_{2|}q_{40} = \frac{1}{50}$
3	$\Pr(K_{40} = 3) = {}_{3|}q_{40} = \frac{1}{50}$
⋮	⋮
49	$\Pr(K_{40} = 49) = {}_{49|}q_{40} = \frac{1}{50}$

Let $W = Min(K_{40}, 2)$. So if $K_{40} = 0$ then $W = 0$, and if $K_{40} = 1$ then $W = 1$. However, if $K_{40} \geq 2$ then $W = 2$. Therefore the probability distribution table for $W = Min(K_{40}, 2)$ is

$W = Min(K_{40}, 2)$	Pr
0	$\frac{1}{50} = \Pr(K_{40} = 0) = q_{40}$
1	$\frac{1}{50} = \Pr(K_{40} = 1) = {}_{1|}q_{40}$
2	$\frac{48}{50} = \Pr(K_{40} \geq 2) = {}_{2}p_{40}$

(a) $E[Min(K_{40}, 2)] = 0 \cdot \frac{1}{50} + 1 \cdot \frac{1}{50} + 2 \cdot \frac{48}{50} = \frac{97}{50} = 1.94.$

(b) $Var(Min(K_{40}, 2)) = Var(W) = E[W^2] - (E[W])^2$. From part (a), $E[W] = 1.94$, and $E[W^2] = 0^2 \cdot \frac{1}{50} + 1^2 \cdot \frac{1}{50} + 2^2 \cdot \frac{48}{50} = \frac{193}{50} = 3.86$. Therefore $Var(Min(K_{40}, 2)) = 3.86 - 1.94^2 = .0964$

17. We can capture the given information in a table as follows

k	q_{x+k}
0	.1
1	.2
2	.3
$\vdots$	$\vdots$
9	1

Therefore

(a) $q_x = .1$

(b) ${}_{1|}q_x = p_x \cdot q_{x+1} = (.9)(.2) = .18$

(c) ${}_{2|}q_x = {}_2p_x \cdot q_{x+2} = p_x \cdot p_{x+1} \cdot q_{x+2} = (.9)(.8)(.3) = .216$

(d) ${}_3p_x = p_x \cdot p_{x+1} \cdot p_{x+2} = (.9)(.8)(.7) = .504$

18. Let $W = Min(K_x, 3)$. Using the logic in Number 16 and mortality assumptions from Number 17, the probability distribution table for $W = Min(K_x, 3)$ is

$W = Min(K_x, 3)$	Pr
0	$\Pr(K_x = 0) = q_x = .1$
1	$\Pr(K_x = 1) = {}_{1\|}q_x = .18$
2	$\Pr(K_x = 2) = {}_{2\|}q_x = .216$
3	$\Pr(K_x \geq 3) = {}_3p_x = .504$

Note that the sum of the probabilities is $.1 + .18 + .216 + .504 = 1$, as it should be. Also, $E[W^2] = 0^2 \cdot (.1) + 1^2 \cdot (.18) + 2^2 \cdot (.216) + 3^2 \cdot (.504) = 5.58$ and $E[W] = 0 \cdot (.1) + 1 \cdot (.18) + 2 \cdot (.216) + 3 \cdot (.504) = 2.124$. Therefore $Var(Min(K_x, 3)) = 5.58 - 2.124^2 = 1.068624$

19. (a) $e_{90} = E[K_{90}] = 0 \cdot (.2) + 1 \cdot (.3) + 2 \cdot (.5) = 1.3$

(b) $Var(K_{90}) = E[(K_{90})^2] - (E[K_{90}])^2$. From part (a), $E[K_{90}] = 1.3$, and since $E[(K_{90})^2] = 0^2 \cdot (.2) + 1^2 \cdot (.3) + 2^2 \cdot (.5) = 2.3$, we get $Var(K_{90}) = .61$

20. (See Video Solution) $e_{20} = 39.5$

21. (See Video Solution) $e_{20} = \frac{91}{8}$

22. We generally have two ways to determine $\overset{o}{e}_x$, the expected value of the continuous random variable, T_x. We can use the statistics definition, $\overset{o}{e}_x = E[T_x] = \int_0^\infty t \cdot f_x(t)\, dt$, or we can use the "integration by parts" formula $\overset{o}{e}_x = \int_0^\infty {}_tp_x \; dt$. Based on the given information for this problem, and this is usually the case, it's easier to use the "integration by parts" formula; namely, $\overset{o}{e}_x = \int_0^\infty e^{-.05t}\; dt = \frac{1}{.05}(1 - e^{-\infty}) = 20$.

23. There are two ways to proceed, based on the two ways to determine $\overset{o}{e}_{20}$ discussed in the previous problem. Let's use the statistics definition first.

Note that ${}_tq_{20}$ is the (cumulative) distribution function of the T_{20} random variable, and so the pfd of T_{20} is $f_{20}(t) = F_{20}'(t) = {}_t\dot{q}_{20} = \frac{1}{80}$ for $0 \le t \le 80$. Implied here is that $f_{20}(t) = 0$ for $t > 80$. You should recognize this as the pdf of a (continuous) uniform distribution over the interval [0,80]. The expected value of a (continuous) uniformly distributed random variable over an interval is just the midpoint of the interval. So $\overset{o}{e}_{20} = 40$. The longer approach is to actually go through the integration. Then we would have $\overset{o}{e}_{20} = E[T_{20}] = \int_0^\infty t \cdot f_x(t) dt = \int_0^{80} t \cdot \frac{1}{80}\, dt + \int_{80}^\infty t \cdot 0\, dt = \frac{t^2}{160}\big|_0^{80} = \frac{80^2}{160} = 40$.

The second method is to use the integration by parts formula; i.e. to integrate the survival function ${}_tp_{20} = 1 - \frac{t}{80}$. Note that if $t > 80$, then $1 - \frac{t}{80}$ yields a negative value, and so once again, implied here is that ${}_tp_{20} = 1 - \frac{t}{80}$ for $0 \le t \le 80$, and ${}_tp_{20} = 0$ for $t > 80$. Then $\overset{o}{e}_{20} = E[T_{20}] = \int_0^\infty {}_tp_{20}\, dt = \int_0^{80}(1 - \frac{t}{80})\, dt = (t - \frac{t^2}{160})\big|_0^{80} = 80 - \frac{80^2}{160} = 40$ as before.

24. This is a basic application of the 1-year recursion formula: $e_x = p_x(1 + e_{x+1})$. With $x = 50$ we have $e_{50} = p_{50}(1 + e_{51}) \Rightarrow 25 = .98(1 + e_{51}) \Rightarrow e_{51} = \frac{25}{.98} - 1 \approx 24.51$.

25. This is another basic application of the 1-year recursion formula: $e_x = p_x(1 + e_{x+1})$. With $x = 60$ we have $e_{60} = p_{60}(1 + e_{61}) \Rightarrow e_{60} = .95(1 + 10) \Rightarrow e_{60} = 10.45$.

26. (See Video Solution) $e_{32} = 33.5$

27. This is another illustration of the 2-year recursion formula for e_x:
$e_x = p_x + {}_2p_x\,(1 + e_{x+2}) = p_x + p_x \cdot p_{x+1}(1 + e_{x+2})$. Therefore

$$4.5 = p_x\left[1 + \frac{8}{9}(1 + 3.5)\right] \Rightarrow p_x = .9$$

28. (See Video Solution) $e_x = p_x + {}_2p_x + {}_3p_x\,(1 + e_{x+3})$.