Solutions to MLC Module 1 Section 2 Exercises

1. (See Video Solution)
   (a) P ≈ .000307877
   (b) P ≈ .000005542

2. The definition of the force of mortality at age $x+t$ is $\mu_{x+t} = \frac{-\,{}_t\dot{p}_x}{{}_tp_x}$. Since ${}_t\dot{p}_x = \frac{d}{dt}[e^{-.05t}] = -.05e^{-.05t}$, then $\mu_{x+t} = \frac{-\,{}_t\dot{p}_x}{{}_tp_x} = \frac{-(-.05e^{-.05t})}{e^{-.05t}} = .05$.

3. Recall that $\frac{d}{dt}[a^t] = a^t \cdot \ln(a)$. Therefore, $\mu_{x+t} = \frac{-\,{}_t\dot{p}_x}{{}_tp_x} = \frac{-(.9)^t \cdot \ln(.9)}{(.9)^t} = -\ln(.9)$.

4. Note that we think of x as constant (a fixed age), and t is the independent variable. That's why the derivative in the definition of μ_{x+t} is with respect to t. Also, μ_{x+t} is the force of mortality at age $x+t$. Technically, μ_{x+t} is the force of mortality for (x) but t years later, and so at age $x+t$.

In this problem, since ${}_tq_x = \frac{t}{100-x}$, then ${}_tp_x = 1 - \frac{t}{100-x} = \frac{100-x-t}{100-x}$. Therefore ${}_t\dot{p}_x = \frac{-1}{100-x}$ and so $\mu_{x+t} = \frac{-\,{}_t\dot{p}_x}{{}_tp_x} = \frac{\left(\frac{1}{100-x}\right)}{\left(\frac{100-x-t}{100-x}\right)} = \frac{1}{100-x-t} = \frac{1}{100-(x+t)}$. We seek the force of mortality at age 50, and so we have $x + t = 50$. Therefore, $\mu_{50} = \frac{1}{100-50} = .02$.

5. We have ${}_t\dot{p}_x = 2\left(\frac{100-x-t}{100-x}\right)^1 \cdot \left(\frac{-1}{100-x}\right)$. Therefore,
$\mu_{x+t} = \frac{-\,{}_t\dot{p}_x}{{}_tp_x} = \frac{-2\left(\frac{100-x-t}{100-x}\right)^1 \cdot \left(\frac{-1}{100-x}\right)}{\left(\frac{100-x-t}{100-x}\right)^2} = \frac{\left(\frac{2}{100-x}\right)}{\left(\frac{100-x-t}{100-x}\right)} = \frac{2}{100-x-t} = \frac{2}{100-(x+t)}$. As in the previous problem, we seek the force of mortality at age 50, and so $\mu_{50} = \frac{2}{100-50} = .04$.

6. We have ${}_t\dot{p}_x = \frac{1}{2}\left(\frac{100-x-t}{100-x}\right)^{\frac{-1}{2}} \cdot \left(\frac{-1}{100-x}\right)$. Therefore,
$\mu_{x+t} = \frac{-\,{}_t\dot{p}_x}{{}_tp_x} = \frac{-\frac{1}{2}\left(\frac{100-x-t}{100-x}\right)^{\frac{-1}{2}} \cdot \left(\frac{-1}{100-x}\right)}{\left(\frac{100-x-t}{100-x}\right)^{\frac{1}{2}}} = \frac{\left(\frac{1/2}{100-x}\right)}{\left(\frac{100-x-t}{100-x}\right)} = \frac{1/2}{100-x-t} = \frac{1/2}{100-(x+t)}$. As in the previous problems, we seek the force of mortality at age 50, and so $\mu_{50} = \frac{1/2}{100-50} = .01$.

7. Note that ${}_tp_0$ is the survival function for a newborn. Since we seek the force of mortality at age 4, we focus on the expression ${}_tp_0 = 1 - \frac{e^t}{100}$. Then ${}_t\dot{p}_0 = -\frac{e^t}{100}$, and so $\mu_t = \frac{-\,{}_t\dot{p}_0}{{}_tp_0} = \frac{\left(\frac{e^t}{100}\right)}{\left(1-\frac{e^t}{100}\right)} = \frac{e^t}{100-e^t}$. Therefore, $\mu_4 = \frac{e^4}{100-e^4}$.

8. Note that since $\mu_{20+t} = \frac{-\,{}_t\dot{p}_{20}}{{}_tp_{20}} = \frac{f_{20}(t)}{{}_tp_{20}}$, then by clearing out the fraction we see that ${}_tp_{20} \cdot \mu_{20+t} = f_{20}(t)$. Therefore, $\int_0^{10} {}_tp_{20} \cdot \mu_{20+t}\, dt = \int_0^{10} f_{20}(t)\, dt = {}_{10}q_{20}$. Then we're given ${}_{10}q_{20} = .05$, and so ${}_{10}p_{20} = .95$.

9. As above ${}_tp_{50} \cdot \mu_{50+t} = f_{50}(t)$, and so $\int_0^{15} {}_tp_{50} \cdot \mu_{50+t} dt = {}_{15}q_{50}$. Therefore we're given ${}_{20}p_{30} = .85$ and ${}_{15}q_{50} = .2$, and we seek ${}_{35}q_{30}$. Using factorization of p's, we have ${}_{35}p_{30} = {}_{20}p_{30} \cdot {}_{15}p_{50} = (.85) \cdot (.8) = .68$, and so ${}_{35}q_{30} = .32$.

10. Note that $\int_{10}^{30} {}_tp_{30} \cdot \mu_{30+t} dt = \int_{10}^{30} f_{30}(t) dt = \Pr(10 < T_{30} < 30) = {}_{10|20}q_{30}$. Since we seek ${}_{30}p_{30}$, we use ${}_{10|20}q_{30} = {}_{10}p_{30} - {}_{30}p_{30}$. Then $\frac{2}{7} = \frac{6}{7} - {}_{30}p_{30}$, and so ${}_{30}p_{30} = \frac{4}{7}$.

11. We seek the probability that a 20-year old survives to age 30. In order to do so, the 20-year old must resist the force of mortality between the ages of 20 and 30. The probability of doing so is $e^{-\int_0^{10} \mu_{20+t}\, dt}$. Notice that we are integrating over the interval from 0 to 10. As the variable of integration, t, ranges from 0 to 10, we are integrating the force of mortality over the ages 20 to 30. This corresponds to the above statement that the 20-year old must resist the force of mortality between the ages of 20 and 30. Therefore, we get ${}_{10}p_{20} = e^{-\int_0^{10} \frac{1}{60-t} dt} = e^{\ln(60-t)|_0^{10}} = e^{\ln\left(\frac{50}{60}\right)} = \frac{5}{6}$.

12. As in the previous problem, to say that the 20-year survives to age 30 is to say that the 20-old resists the force of mortality between the ages of 20 and 30. In this problem, the force is given as μ_t instead of as μ_{20+t}. Using μ_t we represent the probability that the 20-year old resists the force of mortality between the ages of 20 and 30 as $e^{-\int_{20}^{30} \mu_t\, dt}$. Once again, as the variable of integration, t, ranges from 20 to 30, we are integrating the force of mortality over the ages 20 to 30. A common TRAP in this type of problem is to evaluate the expression $e^{-\int_0^{10} \mu_t\, dt}$. However, in this expression, we are integrating the force of mortality over the ages 0 to 10, and so evaluation of this expression actually gives ${}_{10}p_0$. Therefore, the correct answer is ${}_{10}p_{20} = e^{-\int_{20}^{30} \frac{1}{80-t} dt} = e^{\ln(80-t)|_{20}^{30}} = e^{\ln\left(\frac{50}{60}\right)} = \frac{5}{6}$. As we shall later see, it is not a coincidence that we get the same answer as in the previous problem.

13. Don't get confused by the choice of the variable representing the age. In the last problem we used t and in this problem we have x. It doesn't matter; the process is the same. Also, the time and duration values (19 and 17, respectively) may seem like odd choices, but you'll see why they are what they are shortly. We seek the probability that a 19-year old survives to age 36. In order to do so, the 19-year old must resist the force of mortality over the ages 19 to 36. The probability of doing so is

$${}_{17}p_{19} = e^{-\int_{19}^{36} \mu_x\, dx} = e^{-\int_{19}^{36} (100-x)^{-\frac{1}{2}}\, dx} = e^{2(100-x)^{\frac{1}{2}}|_{19}^{36}} = e^{2(\sqrt{64}-\sqrt{81})} = e^{-2}$$

14. In this problem we're given the force of mortality in the μ_{x+t} form. We seek the probability that a 25-year old survives to age 26. In order to do so, the 25-year old must resist the force of mortality over the ages 25 to 26. The probability of doing so is

$$p_{25} = e^{-\int_0^1 \mu_{25+t}\,dt} = e^{-\int_0^1 .1(1-t)\,dt} = e^{-.1\left(t-\frac{t^2}{2}\right)|_0^1} = e^{-.1(.5)} = e^{-.05}$$

15. We seek the probability that a newborn survives to age 0.4. In order to do so, the newborn must resist the force of mortality over the ages 0 to 0.4. The probability of doing so is

$$_{0.4}p_0 = e^{-\int_0^{0.4} \mu_x\,dx} = e^{-\int_0^{0.4} e^{2x}\,dx} = e^{\left(-\frac{1}{2}e^{2x}|_0^{0.4}\right)} = e^{\left(-\frac{1}{2}(e^{0.8}-1)\right)} \approx 0.5418$$

16. In the expression $\int_{71}^{75} \mu_x\,dx$, we are integrating the force of mortality over the ages 71 to 75. We use this expression to evaluate the probability that a 71-year old survives to age 75; namely, $_4p_{71} = e^{-\int_{71}^{75} \mu_x\,dx} = e^{-.107}$.

In the expression $\int_0^5 \mu_{70+t}dt$, notice that as we integrate t over the interval from 0 to 5, we are integrating the force of mortality over the ages 70 to 75. We use this expression to evaluate the probability that a 70-year old survives to age 75;
namely, $_5p_{70} = e^{-\int_0^5 \mu_{70+t}\,dt} = e^{-.189}$.

We seek q_{70}, and we use factorization of p's to get it. We have $_5p_{70} = p_{70} \cdot {}_4p_{71}$ and so $p_{70} = \frac{e^{-.189}}{e^{-.107}} = e^{-.082} \Rightarrow q_{70} = 1 - e^{-.082}$.

17. We generally have $_3p_{x+10} = e^{-\int_{x+10}^{x+13} \mu_t\,dt}$. Notice the argument of the force of mortality has been changed from x to t. This is because we're using x in the limits of integration, and so it would be confusing (and poor notation) to also use x as the integration variable. The integration variable is generally a dummy variable, and so we can change it to whatever we want.

Note that in addition to $_3p_{x+10} = e^{-\int_{x+10}^{x+13} \mu_t\,dt}$, we also have $_3p_{x+10} = e^{-\int_{10}^{13} \mu_{x+t}\,dt}$ and $_3p_{x+10} = e^{-\int_0^3 \mu_{x+10+t}\,dt}$. In each of these three cases, you should notice that as the variable of integration, t, ranges over the corresponding interval of integration, then we are integrating the force of mortality over the ages $x + 10$ to $x + 13$, thus determining the probability of an $x + 10$ year old surviving to age $x + 13$; i.e. $_3p_{x+10}$.

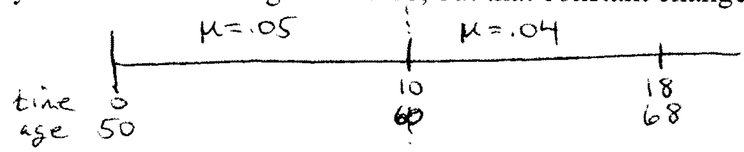
It is easy to see that all three cases in the above paragraph yield the same value when you have a constant force of mortality, since $\int_a^b c\ dt = c \cdot (b - a)$. For all three integrals in the previous paragraph, $b - a = 3$ and $c = \mu_t = .02$, so $_3p_{x+10} = e^{-.02(3)} = e^{-.06}$, regardless of which expression we use.

It is extremely helpful to recognize that when the force of mortality is constant, the value of ${}_np_y$ does not depend on the age, the post-subscript, but only on the duration, the pre-subscript. Therefore, when given a constant force of interest, μ, we often leave off the post-subscript and write ${}_np$ for the probability that the person (or object) under consideration survives n years. Also note that ${}_np = e^{-\mu n} = p^n$ where $p = e^{-\mu}$.

18. Whenever you're given a force of mortality, you should generally think of p's first. So we'll calculate the corresponding p and then use $q = 1 - p$, with appropriate decorations.

For part (a), ${}_4p_{50}$ is the probability a 50-year old survives to age 54. The force of mortality is .05 for ages 50 to 54. As like the previous problem, ${}_4p_{50} = {}_4p = e^{-.05(4)}$. Therefore, ${}_4q_{50} = {}_4q = 1 - {}_4p = 1 - e^{-.2}$.

For part (b), ${}_{18}p_{50}$ is the probability a 50-year old survives to age 68. The force of mortality is constant from ages 50 to 68, but that constant changes at age 60.



So we need to factor ${}_{18}p_{50}$ as ${}_{10}p_{50} \cdot {}_8p_{60}$. Then ${}_{10}p_{50} = {}_{10}p = e^{-.05(10)} = e^{-.5}$ and ${}_8p_{60} = {}_8p = e^{-.04(8)} = e^{-.32}$. Then, ${}_{18}p_{50} = e^{-.5} \cdot e^{-.32} = e^{-.82} \Rightarrow {}_{18}q_{50} = 1 - e^{-.82}$.

19. (See Video Solution)

(a) ${}_{10}q_x^s = 1 - e^{-2}$

(b) ${}_{10}q_x^{ns} = 1 - e^{-1}$

(c) ${}_{10}q_x = 1 - .3e^{-2} - .7e^{-1}$

(d) $\pi \approx 11.5614$

20. (See Video Solution)

(a) ${}_{60}p_0^m = e^{-6}$

(b) ${}_{60}p_0^f = e^{-4.8}$

(c) ${}_{60}p_0 = .5e^{-6} + .5e^{-4.8}$.

(d) $\frac{e^{-6}}{e^{-6}+e^{-4.8}}$

(e) $q_{60} = (1 - e^{-.1}) \cdot \left(\frac{e^{-6}}{e^{-6}+e^{-4.8}}\right) + (1 - e^{-.08}) \cdot \left(\frac{e^{-4.8}}{e^{-6}+e^{-4.8}}\right)$

21. (a) ${}_5p_x = e^{-5\mu}$

(b) We now think of μ as a discrete random variable, taking on the value .02 with probability 30%, and the value .03 with probability 70%. We have $\left({}_5p_x \mid \mu\right) = e^{-5\mu}$ and the law of total probability yields

$$ {}_5p_x = E\left[{}_5p_x \mid \mu\right] = E[e^{-5\mu}] = e^{-5(.02)}(.3) + e^{-5(.03)}(.7) = .3e^{-.1} + .7e^{-.15} $$

(c) As in part (b), we have that μ is a random variable and we have $\left({}_5p_x \mid \mu\right) = e^{-5\mu}$. Now μ is a continuous random variable, uniformly distributed on [0.01,0.02]. Therefore, its density function is $\frac{1}{.02-.01} = 100$. Then the law of total probability gives

$$ {}_5p_x = E\left[{}_5p_x \mid \mu\right] = E[e^{-5\mu}] = \int_{0.01}^{0.02} e^{-5\mu} \cdot 100\, d\mu = 20 \cdot (e^{-.05} - e^{-.1}) $$

22. $p_x^f = e^{-\int_0^1 \mu_{x+t}^f\, dt} = e^{-\int_0^1 (\mu_{x+t}^m - .02)\, dt} = e^{-\int_0^1 \mu_{x+t}^m\, dt + \int_0^1 .02\, dt} = e^{-\int_0^1 \mu_{x+t}^m\, dt} \cdot e^{\int_0^1 .02\, dt}$

Recognize the first factor in the last expression as p_x^m, and the second factor equals $e^{.02}$. Therefore, $p_x^f = p_x^m \cdot e^{.02} = .95e^{.02}$

23. ${}_{10}p_x^m = e^{-\int_0^{10} \mu_{x+t}^m\, dt} = e^{-\int_0^{10}\left(\mu_{x+t}^f + k\right) dt} = e^{-\int_0^{10} \mu_{x+t}^f\, dt} \cdot e^{-\int_0^{10} k\, dt} = {}_{10}p_x^f \cdot e^{-10k}$.

Therefore, $0.64 = 0.74e^{-10k} \Rightarrow k = \frac{-1}{10} \cdot \ln\left(\frac{64}{74}\right)$

24. ${}_kp_x^s = e^{-\int_0^k \mu_{x+t}^s\, dt} = e^{-\int_0^k (1.1\mu_{x+t}^{ns})\, dt} = e^{-1.1\int_0^k \mu_{x+t}^{ns}\, dt} = \left(e^{-\int_0^k \mu_{x+t}^{ns}\, dt}\right)^{1.1} = \left({}_kp_x^{ns}\right)^{1.1}$

Therefore, ${}_kp_x^s = (.75)^{1.1}$.

25. ${}_kp_x^{ns} = e^{-\int_0^k \mu_{x+t}^{ns}\, dt} = e^{-\int_0^k (c \cdot \mu_{x+t}^s)\, dt} = e^{-c\int_0^k \mu_{x+t}^s\, dt} = \left(e^{-\int_0^k \mu_{x+t}^s\, dt}\right)^c = \left({}_kp_x^s\right)^c$

Therefore, $0.6 = (0.54)^c \Rightarrow c = \frac{\ln(0.6)}{\ln(0.54)}$