Review of Algebra, Calculus, and Counting Techniques

Set Theory – The typical question can be solved using Venn Diagrams, which we'll illustrate by example.

Inverse Functions – The inverse of the function f is denoted by f^{-1} and is the function for which $f\left(f^{-1}(x)\right) = x = f^{-1}\left(f(x)\right)$. E.g (For example) if $f(x) = 2x+1$ then $f^{-1}(x) = \frac{x-1}{2}$ since $f\left(f^{-1}(x)\right) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x$ and $f^{-1}\left(f(x)\right) = f^{-1}(2x+1) = \frac{(2x+1)-1}{2} = x$.

Finding f^{-1}: Given a function f, we introduce another variable y and write $y = f(x)$. Then solve for x in terms of y. This gives the expression that defines $f^{-1}(y)$; of course if we're looking for $f^{-1}(x)$ then we'll need to replace y by x. This process is illustrated by an example.

<u>Note</u>: Although not all functions have inverses, the question of which functions do have inverses is not tested on Exam P.

Quadratic Formula – The solutions of the general quadratic equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Go to the *formula sheet* tab of your binder and write in this formula.

Derivatives – Of course you should know all of the basic derivative rules, other than derivatives of trigonometric functions. The chain rule is very important and so we discuss this rule further. It is important to always keep in mind the variable which we are taking a derivative with respect to. E.g. suppose we are given $y = h(w)$. The following all denote the derivative of *y* (or *h*) with respect to *w*: $\frac{dy}{dw} = \frac{dh}{dw} = h'(w)$. Now, if w is in turn a function of x (thus making y a function of x as well) then by the chain rule, $\frac{dy}{dx} = \frac{dh}{dw} \cdot \frac{dw}{dx} = h'(w) \cdot w'(x)$. Note that the "prime" notation can be ambiguous.

Partial Derivatives – If we are given a function $f(x,y)$ of two variables, then we can take the "partial" derivative of f with respect to one of the variables by treating the other variable as a constant. For example, we take the partial derivative of f with respect to y, denoted $\frac{\partial f}{\partial y}$ or f_y, by treating x as a constant and using our ordinary derivative rules. The notation and method for calculating higher order partial derivatives are illustrated with an example.

Integration – You should also know all of the basic integration rules, other than integration of trigonometric functions, and be proficient with the substitution method.

The geometric interpretation of the definite integral $\int_a^b f(x)dx$ is that it represents the area under the curve $y = f(x)$ over the interval $[a,b]$. The Fundamental Theorem of Calculus (FTC) relates the concepts of differentiation and integration. It states that in order to compute $\int_a^b f(x)dx$ we need only to find an anti-derivative of $f(x)$, which we'll denote $F(x)$, and then calculate the difference $F(b) - F(a)$. Additionally, if we define $F(x) = \int_a^x f(t)dt$ then $F'(x) = f(x)$. We'll illustrate the FTC with an example.

Double Integrals – These will show up quite a bit later in the seminar. We'll illustrate this concept by examples.

Arithmetic Progressions – These are progressions that are defined by declaring that the *difference* between consecutive terms is constant. So the terms of an arithmetic progression look like

a, $a+d$, $a+2d$, $a+3d$, … (d = difference between consecutive terms)

I remember the formula for the value of an arithmetic sum by saying it as follows:

"The sum equals the average of the first and last term, times the number of terms."

Geometric Progressions – These are progressions that are defined by declaring that the *ratio* of consecutive terms is constant. So the terms of a geometric progression look like

a, ar, ar^2, ar^3, … (*a* = first term and *r* = ratio between consecutive terms)

A formula for the sum of the first *n* terms of a geometric progression is

$$s = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}$$

I find it easiest to remember this formula by saying it as follows:

"The sum equals the first term minus the first omitted term, all divided by one minus the ratio."

The following derivation of the above formula is worth seeing, as the "trick" used in the derivation can be applied in other contexts.
Let *s* denote the sum and we will solve for *s*. Then $s = a + ar + ar^2 + ... + ar^{n-1}$.
Multiply both sides of this equation by the common ratio *r*, subtract the resulting equation from the original, and then solve for *s*.

$$s = a + ar + ar^2 + ... + ar^{n-1}$$
$$\underline{s \cdot r = ar + ar^2 + ... + ar^{n-1} + ar^n}$$
$$s \cdot (1 - r) = a - ar^n = a \cdot (1 - r^n) \Rightarrow s = a \cdot \frac{1 - r^n}{1 - r}$$

The above sum is the sum of the first *n* terms. If we sum *all* of the terms, then we get a geometric series whose sum is $a + ar + ar^2 + ... = \frac{a}{1 - r}$, as long as $-1 < r < 1$ (which will be the case for us).

Note: In words, this formula reads: **"The sum equals the first term, divided by one minus the ratio."** The point here is that in words, you have the same formula. *There are no omitted terms in the series!*

We illustrate these formulas with an example.

Fundamental Counting Principle (FCP) – The FCP states that if there are *m* ways of performing task 1 and *n* ways of performing task 2, then there are *mn* ways of performing the job consisting of tasks 1 and 2. This can easily be generalized to more than 2 tasks.

We illustrate the FCP with an example.

Permutations – We use permutations to count the number of ways we can choose a set of *k* objects from a collection of *n* distinct objects, where repetitions are not allowed and the order of the *k* objects does matter. We denote this number of ways by ${}_nP_k$, and the formula for ${}_nP_k$ is:

$${}_nP_k = \frac{n!}{(n-k)!} = n\cdot(n-1)\cdot(n-2)\cdots(n-(k-1))$$

Combinations – We use combinations to count the number of ways we can choose a set of *k* objects from a collection of *n* distinct objects, where repetitions are not allowed and the order of the *k* objects does not matter. We denote this number by ${}_nC_k$, and the formula for ${}_nC_k$ is:

$${}_nC_k = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-(k-1))}{k!}$$

The following flow chart will help determine which counting principle should be used.

Q1: Are repetitions allowed?
- A1 = yes; Use FCP
- A1 = no; Q2: Does order matter?
  - A2 = yes; Use Permutations or FCP
  - A2 = no; Use Combinations

Distinguishable Permutations – Given n_1 identical objects of Type 1 and n_2 identical objects of Type 2, then the number of distinguishable orderings (permutations) of all $n = n_1 + n_2$ objects is $\frac{n!}{n_1!\,n_2!}$. Note that we can generalize to more than 2 object types.

We illustrate counting techniques with several examples.