Basic and Conditional Probability

Probability Concepts – The collection of all possible outcomes when an experiment is performed is called a probability space, denoted $S$. An event is a subset of the probability space. Events are usually denoted by capital letters ($A$, $B$, etc.) Each event has a probability of occurring, which is essentially how big the event is in comparison to the experiment. The notation is that the probability of event $A$ is denoted $\Pr(A)$. We illustrate the notation and basic probability concepts using a simple example.

Mutually Exclusive Events – Event $A$ and $B$ are mutually exclusive if there is no outcome common to both $A$ and $B$, i.e. $A \cap B = \emptyset$.

DeMorgan’s Laws – These are often tested rules that relate the set operations; union, intersection, and complements,

\[
(A \cup B)' = A' \cap B' \\
(A \cap B)' = A' \cup B'
\]

Note: DeMorgan’s Laws can be generalized to more than two sets. E.g.

\[
(A \cup B \cup C)' = A' \cap B' \cap C' \\
(A \cap B \cap C)' = A' \cup B' \cup C'
\]

Other often tested set operations and probability rules:
(These rules can also be generalized)

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
4. $\Pr(A') = 1 - \Pr(A)$
5. $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$ (will be used often)

We illustrate these facts by examples.
**Conditional Probability** – denoted by \( \Pr(B \mid A) \), is the probability that event \( B \) occurs given that event \( A \) has occurred. The formula for \( \Pr(B \mid A) \) is

\[
\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)}
\]

Note that we can rewrite the above formula as

\[
\Pr(A \cap B) = \Pr(B \mid A) \cdot \Pr(A) = \Pr(A \mid B) \cdot \Pr(B)
\]

Together with Rule 5 from above, we get the following often used result

\[
\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B') = \Pr(A \mid B) \cdot \Pr(B) + \Pr(A \mid B') \cdot \Pr(B')
\]

We illustrate how to use this formula with an example.

**Bayes’ Rule (Theorem)** – This is an often tested technique used to solve a certain type of problem. We will be asked to find \( \Pr(B \mid A) \) and we do so by first using the above formula to find \( \Pr(A) \). Notice that \( \Pr(A) \) is a sum of terms, one of which is \( \Pr(A \cap B) \). Therefore we have all the information needed to calculate \( \Pr(B \mid A) \). We illustrate how to use this theorem with a couple of examples.

**Independent Events** – By definition, events \( A \) and \( B \) are independent events if \( \Pr(A \cap B) = \Pr(A) \cdot \Pr(B) \). This is equivalent to the statement that \( \Pr(B \mid A) = \Pr(B) \). More generally, for any events \( A, B, \) and \( C \), we have

\[
\Pr(A \cap B \cap C) = \Pr(A \cap B \mid C) \cdot \Pr(C) = \Pr(A \mid B \cap C) \cdot \Pr(B \mid C) \cdot \Pr(C),
\]

whereas if \( A, B, \) and \( C \) are mutually independent then \( \Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C) \)

Other conditional probability and independence rules:

(这些规则可以推广)

1. \( \Pr(A \cup B \mid C) = \Pr(A \mid C) + \Pr(B \mid C) - \Pr(A \cap B \mid C) \)

2. \( \Pr(A' \mid B) = 1 - \Pr(A \mid B) \)

We illustrate how to use these formulas with an example.