March 30, 2013

Directions: This practice test is designed to help your preparation for Test 2, and in particular the part concerning ODEs. Test 2 materials will include

(i) The grand theorem.
(ii) $e$Values and $e$Vectors.
(iii) Links with 2nd order ODEs.

You may turn in this assignment for extra credit. Please stop by my office hours if you have questions. Make sure to show me your work and explain the steps that lead to your answer. State any theorem you use.

In this Test preparation, we study the behavior of harmonic oscillators. These appear for instance in electrical engineering when studying RLC circuits or in Newtonian mechanics. They arise when solving the homogeneous second order ODE:

$$\frac{d^2y}{dt^2} - p \frac{dy}{dt} + qy = 0.$$  \hspace{1cm} (1)

This second order ODE is equivalent to the system

$$\frac{d}{dt} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} z(t) \\ y(t) \end{bmatrix},$$  \hspace{1cm} (2)

where $z(t) = y'(t)$. We have seen in class that the solutions of Equation (2) are of the form $\alpha X_1 e^{\lambda_1 t} + \beta X_2 e^{\lambda_2 t}$. \(\alpha\) and \(\beta\) are given by the initial data of the problem (the initial conditions), \(\lambda_1\) and \(\lambda_2\) are the eigenvalues of the matrix \(A\) and \(X_1\) and \(X_2\) are the corresponding eigen vectors. We propose here to study two examples of the possible solutions of the ODE (1).

**Question 1** Assume that the values of the coefficients are $p = 2$, $q = -3$.

1. Find the characteristic polynomial $p$ of the matrix $A$.
2. Solve the equation $p(\lambda) = 0$ and deduce its eigen values \(\lambda_1\) and \(\lambda_2\).
3. Find the corresponding eigen vectors $X_1$ and $X_2$.
4. Deduce that the solution of Eq. (1) with $p = 2$, $q = -3$ is given by

$$y(t) = \alpha e^{3t} + \beta e^{-t}$$

for two real numbers $\alpha$ and $\beta$. 

Instructor: Pierre GARREAU
(5) Discuss the solution $y(t)$: what is the behavior of $y$ as time passes by? Is this something you would expect from a physical system? Considering that the solution goes to zero 'quickly', we note in this case that the 'damping' term $py'(t)$ dominates.

**Question 2** Assume now that the values of the coefficients are $p = -2$, $q = 3$.

(1) Show that the characteristic polynomial of $A$ can be written
\[ p(x) = (x + 1)^2 + 2. \]

(2) Deduce that the eigen values of $A$ are complex and equal to $\lambda_1 = -1 - i\sqrt{2}$ and $\lambda_2 = -1 + i\sqrt{2}$.

(3) Show that the products $(-1 + i\sqrt{2})(-1 - i\sqrt{2})$ and $(1 + i\sqrt{2})(-1 + i\sqrt{2})$ are equal to 3 and $-3$. Deduce that the RREF of the matrices $A - \lambda_1 I$ and $A - \lambda_2 I$ are given by
\[ RREF(A - \lambda_1 I) = \begin{bmatrix} 1 & -(-1 - i\sqrt{2}) \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad RREF(A - \lambda_2 I) = \begin{bmatrix} 1 & -(-1 + i\sqrt{2}) \\ 0 & 0 \end{bmatrix}. \]

(4) Deduce from the previous question that the eigen vectors corresponding to $\lambda_1$ and $\lambda_2$ are the vectors
\[ X_1 = \begin{bmatrix} 1 - i\sqrt{2} \\ 1 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 1 + i\sqrt{2} \\ 1 \end{bmatrix}. \]

(5) Conclude by writing the solution of the ODE:
\[ y(t) = e^{-t} \left( \alpha e^{it\sqrt{2}} + \beta e^{-it\sqrt{2}} \right), \quad (3) \]

for two real numbers $\alpha$ and $\beta$.

(6) We are given that the initial conditions are $y'(0) = 1$ (initial velocity) and $y(0) = 0$ (initial position). Use that information to write
\[ \alpha = \frac{1}{2t\sqrt{2}}, \quad \text{and} \quad \beta = -\frac{1}{2t\sqrt{2}}. \]

(7) Show that the solution of this problem is
\[ y(t) = \frac{e^{-t}}{\sqrt{2}} \sin \left( t\sqrt{2} \right). \quad (4) \]

Since the solution goes to zero as $t$ goes to infinity, we still observe damping. However, the $\sin$ function produces oscillations. This is what is called a damped oscillator. (Hint: recall Euler's identity from trigonometry).