Models for Quantitative Portfolio Management
Part 2: Black Litterman and Meucci’s Model

Pierre GARREAU

Florida State University

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Credit Portfolio


Managers' decisions based on Morning signals, News and Intraday Info.

Need to Specify views in a Markowitz type Framework.
Framework

Choice of a fully invested portfolio \( h = [h_1, h_2, \ldots, h_N]^T \) in a universe of \( N \) securities.

*Location* and *Dispersion* measure are *Mean* \( f \) and *Variance* \( \Sigma \).

The investors are Risk Averse with utility function

\[
U(h) = h^T f - \lambda h^T \Sigma h.
\]

\[
h = \arg\max_{h \in \mathcal{H}} U(h)
\]

where

\[
\mathcal{H} = \left\{ \sum h_i = 1, h > c, \frac{h_r^T f}{\sigma_B^2} < q \right\}
\]
Figure: Efficient frontier for a universe of 5 Stocks: Exxon, City, Boeing, Ford, SnP - November 2008
Figure: Efficient frontier for a universe of 5 Stocks: Exxon, City, Boeing, Ford, SnP - November 2008 - Shortsell 20% - Perturbation
Limitations

The risk premium for positive deviations is the same as negative deviations: Approximation of Investors’ satisfaction
Optimizer overweights high expected returns and negatively correlated securities.
Optimizer subject to estimation errors. Relevance of the risk aversion parameter?
Covariances estimation.
The Market The market is represented as a random variable $X$.

$$f_X \sim \mathcal{N}(\mu, \sigma)$$

The market distribution is affected by estimation risk.

The Manager’s selection The manager assesses the outcome of the market $V$ and expresses a conditional distribution $V|x$ or $V|g(x)$. This allows to get

$$f_{X|V}(x \mid v) = \frac{f_{V|g(x)}(v \mid x)f_X(x)}{\int f_{V|g(x)}(v \mid x)f_X(x) \, dx}$$

(1)

The views are specified through a Pick Matrix

$$g(x) = PX$$
Distribution of Views  Evaluation of the manager's track record

\[ V_{|P_x} \sim \mathcal{N}(P_x, \Omega) \]

where

\[ \Omega = \left( \frac{1}{c} - 1 \right) P \Sigma P^T \]

The Manager’s opinion  Asked target views on his pick : \( v \)
The Black-Litterman Model (1990)

The Black-Litterman distribution is normal

\[ X|\nu \sim \mathcal{N}(\mu_{BL}, \Sigma_{BL}) \]

where the expected value and covariance matrix are expressed as

\[ \mu_{BL} = \mu + \Sigma P (P \Sigma P^T + \Omega)^{-1} (\nu - P \mu) \]  \hspace{1cm} (2)

\[ \Sigma_{BL} = \Sigma - \Sigma P^T (P \Sigma P^T + \Omega)^{-1} P \Sigma \]  \hspace{1cm} (3)
Portfolio and Confidence

Full confidence in the manager’s view : $c = 1$

$\Omega \to 0 \Rightarrow \left\{ \begin{array}{c} \mu_{BL} = \mu + \Sigma P (P \Sigma P^T)^{-1} (\nu - P \mu) \\ \Sigma_{BL} = \Sigma - \Sigma P^T (P \Sigma P^T)^{-1} P \Sigma \end{array} \right.$

Conditional Model $PX = \nu$

Null confidence in the manager’s view : $c = 0$

$\Omega \to \infty \Rightarrow \left\{ \begin{array}{c} \mu_{BL} = \mu \\ \Sigma_{BL} = \Sigma \end{array} \right.$

Market Model : Markowitz Portfolio
Investment Universe is composed of 5 Stocks: Aegon, Air Liquide, Allianz, Alstom, Arcelor Mittal.

Covariance Matrix estimated with the method of moments.

\[
\Sigma = \begin{pmatrix}
25, 57\% & -1, 94\% & 5, 68\% & 2, 16\% & 4, 33\% \\
-1, 94\% & 3, 59\% & -0, 28\% & 0, 46\% & 0, 50\% \\
5, 68\% & -0, 28\% & 8, 42\% & 1, 53\% & 1, 75\% \\
2, 16\% & 0, 46\% & 1, 53\% & 13, 37\% & 5, 32\% \\
4, 33\% & 0, 50\% & 1, 75\% & 5, 32\% & 20, 15\%
\end{pmatrix}
\]
Returns are given by the characteristic portfolio of weights
\[ h_w = \sigma_w^2 \Sigma^{-1} w \]

\[
\mu = \begin{pmatrix}
9, 63% \\
1, 39% \\
7, 83% \\
7, 15% \\
13, 81%
\end{pmatrix}
\quad w = \begin{pmatrix}
7, 31% \\
20, 97% \\
34, 96% \\
10, 33% \\
26, 43%
\end{pmatrix}
\]
The targets of the views are different from the impled returns and chosen to show extreme cases.

\[
v = \begin{pmatrix}
-3, 00\% \\
8, 50\% \\
-3, 50\% \\
11, 00\% \\
12, 50\%
\end{pmatrix}
\]
Markowitz’s Portfolio
No confidence in the manager

(a) Black-Litterman $c = 0$

(b) Black-Litterman $c = 1$

Equilibrium portfolio.
Pocket for Air Liquide is increased since 8.5% return and volatility only 3.59% with low or negative correlation with other stocks.
Limitations

Non-Normal Markets and the problem of Asset Correlation.
Express Views on more than just returns.
Stress Testing.
Correlation of Views?
The Market The market is represented as an \( d \)-dimensional random variable \( \mathbf{X} \) and will be specified thanks to its cdf

\[
\mathbf{X} \sim F_{\mathbf{X}}
\]

The \textit{apriori} Market \( \mathcal{M} \) is represented by \( J \) simulations of \( \mathbf{X} : J \times d \)
The Views

The manager expresses an opinion on $K \leq d$ securities so that

$$V = \tilde{P} X$$

where $\tilde{P}$ is the completed and invertible matrix.

Each view is specified individually and represented by its cdf

$$\tilde{F}_k(v) = \mathbb{P}(V_k \leq v) \quad k = 1, \ldots, K$$

Manager’s view is different from the Market Implied Distribution: COP approach
Posterior Marginal CDF

The posterior marginal CDF of each security is given by a linear combination:

\[ \tilde{F}_k(v) = c_k \hat{F}_k(v) + (1 - c_k) F_k(v) \quad k = 1, \ldots, K \]

where \( c_k \) represents the confidence level in view \( k \).

In order to get the joint distribution, we need to estimate the market implied copula.
Copula A d-dimensional Copula C is a d-dimensional distribution function on $[0, 1]^d$ with standard uniform marginal distributions.

Sklar’s Theorem Every cumulative distribution function $F$ with margins $F_1, \ldots, F_d$ can be written as

$$F(v_1, \ldots, v_d) = C(F_1(v_1), \ldots, F_d(v_d))$$

with $C$ unique if the margins are continuous.

Kendall’s Tau For bivariate random vectors $(X, Y)$ Kendall’s Tau is expressed as

$$\tau(X, Y) = \mathbb{E}\left(\text{sign}(X - \tilde{X})(Y - \tilde{Y})\right)$$

where $(\tilde{X}, \tilde{Y})$ is a second independent pair with the same distribution.
Joint Posterior Distribution The market implied copula is then

\[ C^d = (F_1^{-1}(V_1), \ldots, F_d^{-1}(V_d))' \]

The joint posterior distribution of the views is given by:

\[
\begin{pmatrix}
    V_1 \\
    \vdots \\
    V_d
\end{pmatrix}
\overset{d}{=} \begin{pmatrix}
    \tilde{F}_1^{-1}(F_1(V_1)) \\
    \vdots \\
    \tilde{F}_d^{-1}(F_d(V_d))
\end{pmatrix}
\]

Posterior Distribution of the Market We need to apply backward the definition of the views to get \( X \sim \tilde{f}_M \):

\[ X \equiv \tilde{P}^{-1}V \]

where the remaining \( d - K \) entries of \( V \) remain unchanged.
Market Representation

The market is represented as an $d$-dimensional random variable $X \sim T_d(\nu, \mu, \Sigma, \gamma)$

$$X \sim \mu + \gamma W + \sqrt{W} Z$$

with

$Z \sim (0, \Sigma)$ Multivariate Gaussian random variable with 0 mean and covariance matrix $\Sigma$.

$W \sim \text{Inv-Gamma}(\nu/2, \nu/2)$ Inverse Gamma random variable.

$Z \perp W$.

Closed formula for $f_{\nu,\mu,\Sigma,\gamma}$ and $C_{\nu,\mu,\Sigma,\gamma}$
Views Representation

The manager has a view on each of the stocks, and represents it with a target window so that

$$\widehat{F}_k(v) = \begin{cases} 
0 & v \leq a_k \\
\frac{v - a_k}{b_k - a_k} & v \in [a_k, b_k] \\
1 & v \geq b_k
\end{cases}$$
Maximum Likelihood

Taking a set of data that is independent and identically distributed, the log-likelihood function $\mathcal{L}$ of the vector parameter $\theta$ is defined as

$$\mathcal{L}(\theta) = \sum_{j=1}^{J} \log f_\theta(x_j)$$

Estimation of the location and skewness parameters for the marginals

$$(\mu_i, \gamma_i) = \arg\max_{\mu, \gamma} \mathcal{L}_X(\mu, \gamma; r_i^1, \ldots, r_i^J) \quad i = 1, \ldots, d$$

Estimation of the degree of freedom for the Copula

$$\nu^* = \arg\max_\nu \mathcal{L}_C(\nu, \Sigma, \gamma; \hat{u}_i^1, \ldots, \hat{u}_i^J)$$
From the Market distribution $X_{J \times d}$, estimate each empirical cdf by

$$\hat{F}_i(x) = \frac{1}{J + 1} \sum_{j=1}^{J} 1\{X_{j,i} \leq x\} \quad i = 1, \ldots, d$$

Form the pseudo-sample from the copula

$$(\hat{u}_{j,1}, \ldots, \hat{u}_{j,d}) = (\hat{F}_1(X_{j,1}), \ldots, \hat{F}_d(X_{j,d}))$$

Can estimate the dispersion matrix thanks to Kendall’s tau

$$\Sigma_{j,i} = \sin \left( \frac{\pi}{2} \hat{\tau}(X_j, X_i) \right)$$

$$\hat{\tau}(X_j, X_i) = \left( \frac{2}{J} \right)^{-1} \sum_{1 \leq k \leq l \leq J} \text{sign}(X_{l,j} - X_{k,j})(X_{l,i} - X_{k,i})$$
Generate a Multivariate Skewed-t distribution $\mathcal{T}_d(\nu, 0, P, \gamma)$ from the Normal Mixture. 

Return

$$U = (\mathcal{T}_{\nu, \gamma_1}(X_1), \ldots, \mathcal{T}_{\nu, \gamma_d}(X_d))'$$

Figure: $J = 10,000 - \rho = 0.8 - \nu = 5.5$
Calibration of AEGON’s daily returns

(a) Likelihood function, $\nu$ on x-axis for several value of $\gamma_1$

(b) $\mu_1 = -1$, $\sigma_1 = 1.72$, $\gamma_1 = 0.64$, $\nu = 6$
Extensions

Need a relevant Risk Measure to perform an optimization on the a posteriori Market: $\mathbb{E}[X|X < q]$.
More robust Algorithm for Calibration: EM Algorithm.
Possible Extension to Levy Copulas.