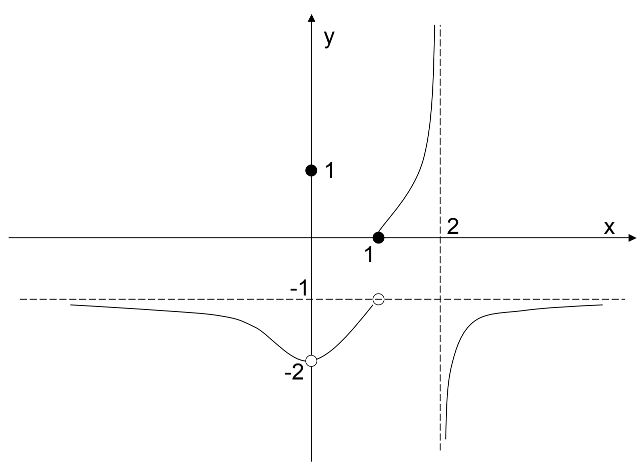
1. Chapter 1 Section 7: Review of Continuity

Definition 1.1 (Intuitive idea used in algebra based on graphing)**.** *Let f be a function and let I be an interval (open, close, or mixed). f is* **continuous on the interval** *I if the graph of $y = f(x)$ can be drawn over the interval without lifting your pencil.*

f is **discontinuous at** *$x = a$, or a* **is a discontinuity of** *f if you have to pick up your pencil as you pass over $x = a$ while tracing over the graph of $y = f(x)$.*

Example 1.1 (Practice: 1.7 Text Problem 1)**.** *Determine the discontinuities. Determine the largest intervals on which the function graphed below is continuous.*



Remarks 1.1.

- *Note that at a discontinuity a small change in x can result in large changes in y.*
- *Use what you know about functions from algebra and trigonometry when considering questions about continuity. Most of the functions that you have worked with are continuous where it is defined.*
- *If a is* **not** *is the domain of f, then f* **cannot** *be continuous at that value.*

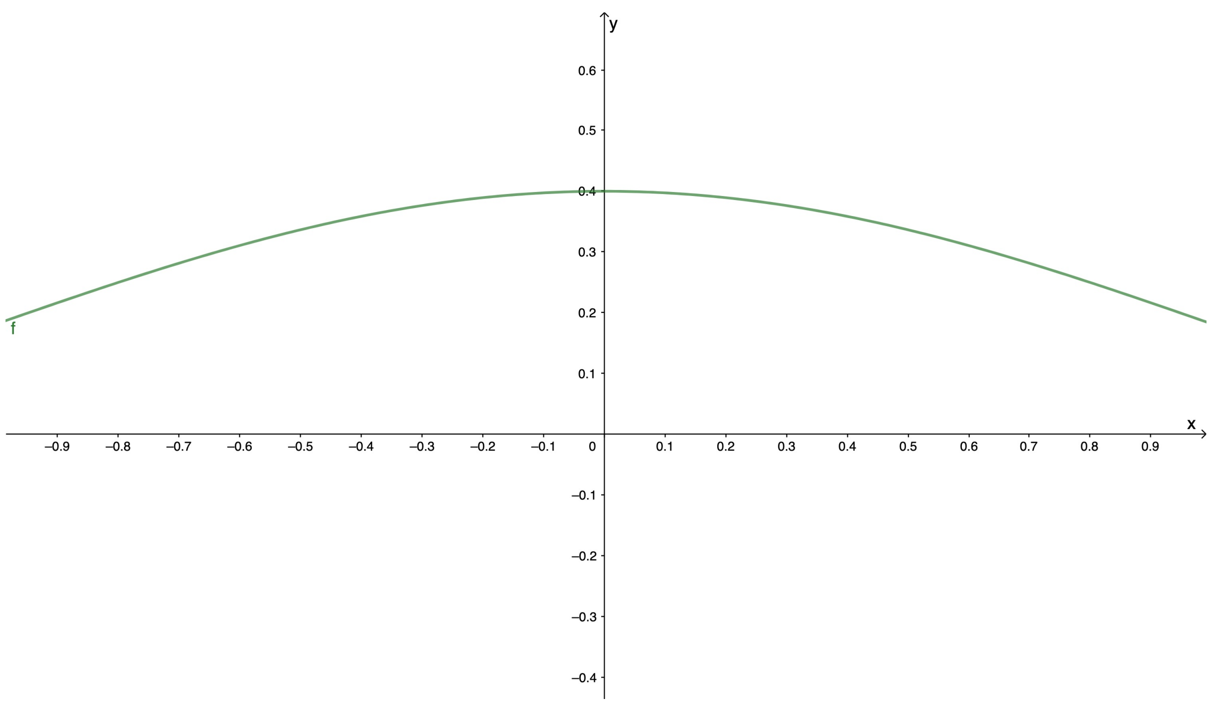
Example 1.2. *Determine the discontinuities. Determine the largest intervals on which the function below is continuous.* $f(x) = \dfrac{x^2 + 3x - 4}{(x+4)(x+1)^2}$

2. Limits

Definition 2.1 (intuitive definition)**.** *The* **limit of** $f(x)$**, as** x **approaches** a**, equals** L *means that as* x *gets arbitrarily close to the value* a *(but not actually equal to* a*), the value of* $f(x)$ *approaches the value* L*. This is also written*

$$\lim_{x \to a} f(x) = L$$

Example 2.1. *Consider the function* $f(x) = \dfrac{\sin 2x}{5x}$ *and find* $\lim_{x \to 0} f(x)$*.*



x	$\frac{\sin x}{5x}$	x	$\frac{\sin x}{5x}$
0.5	0.336588394	-0.5	0.336588394
0.1	0.397338662	-0.1	0.397338662
0.01	0.399973334	-0.01	0.399973334
0.001	0.399999733	-0.001	0.399999733
0.0001	0.399999997	-0.0001	0.399999997
0.00001	0.4	-0.00001	0.4

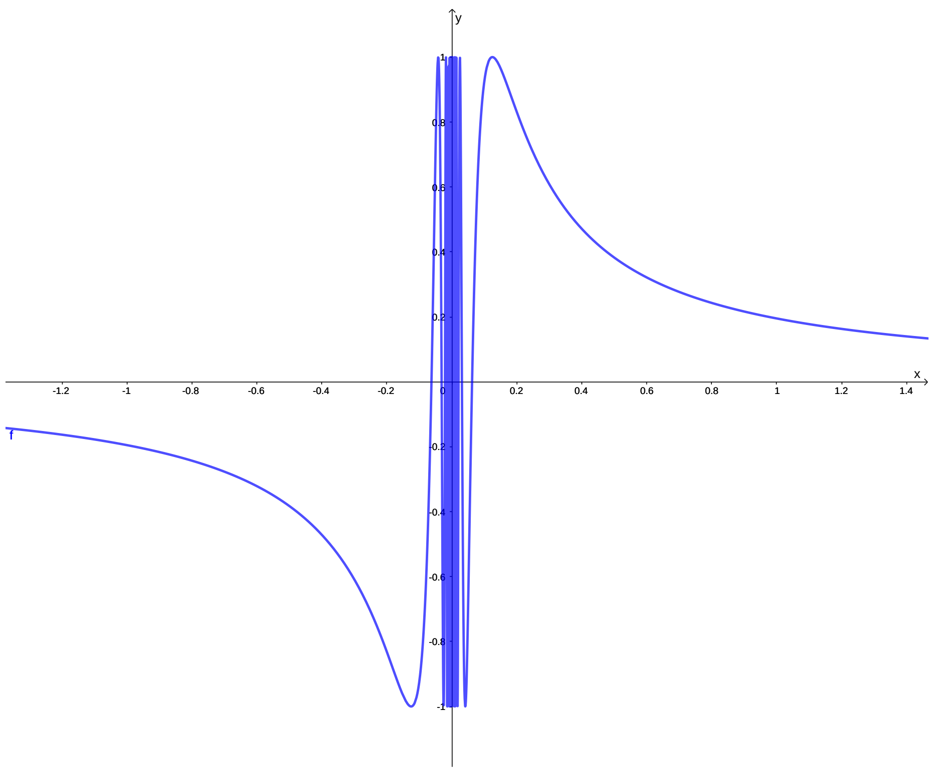
Example 2.2. *Consider the function* $f(x) = \sin \frac{\pi}{16x}$. *Use the table of values below to guess the value of* $\lim_{x \to 0} f(x)$.

x	$\sin(\frac{\pi}{16x})$	x	$\sin(\frac{\pi}{16x})$
0.5	0.382683432	-0.5	-0.382683432
0.1	0.923879533	-0.1	-0.923879533
0.01	0.707106781	-0.01	-0.707106781
0.001	1	-0.001	-1
0.0001	$2.5776E$-13	-0.0001	$-2.5776E$-13
0.00001	$-2.12285E$-12	-0.00001	$2.12285E$-12

Example 2.3 (Practice: 1.7 WP Homework Question 2, Text Problems 6 and 7)**.** *Consider the function* $f(x) = \sin \frac{\pi}{16x}$ *again. Use the table of values below to guess the value of* $\lim_{x \to 0} f(x)$.

n	$x = \frac{1}{8(2n+1)}$	$\sin(\frac{\pi}{16x})$
10	0.005952381	1
11	0.005434783	-1
1000	6.24688E-05	1
1001	6.24064E-05	-1
1000000	6.25E-08	1
1000001	6.24999E-08	-1

Desmos: https://www.desmos.com/calculator/bcf1n2u2be



3. Careful Definition of Continuity

Definition 3.1 (The carefully thought-out calculus version based on limits)**.**

(1) A function, f, is **continuous** *at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.*

(2) A function, f, is **continuous on the interval** (a, b) *if f is continuous at every value in (a, b).*

(3) A function, f, is **discontinuous** *at $x = a$ if f is not continuous at $x = a$. Furthermore, $x = a$ is called a* **discontinuity** *of f.*

(4) Many discontinuities may be classified as a **removable**, **jump**, *or* **infinite** *discontinuity.*

Example 3.1 (Practice: 1.7 Text Problem 18)**.** *Determine the discontinuities and the largest open intervals on which the function is continuous in $(-10, \infty)$.*

$$f(x) = \begin{cases} -1 & \text{if } -10 < x < -2 \\ 2x + 3 & \text{if } -2 \le x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Example 3.2 (Practice 1.7 WP Homework Question 6, Text 34). *For what value of the constant k is the function f continuous on $(-\infty, \infty)$?*

$$f(x) = \begin{cases} kx^2 + 4 & \text{if } x < 2 \\ x^3 - kx & \text{if } x \geq 2 \end{cases}$$

Use the slider on the Desmos graph https://www.desmos.com/calculator/bkdkwyvxjw to estimate k. Then use algebra to find the exact value of k.

4. Intermediate Value Theorem

Theorem 4.1 (The Intermediate Value Theorem). *Let f be continuous on the interval $[a, b]$. If k is a number between $f(a)$ and $f(b)$, then there is at least one number c in the interval (a, b) with $f(c) = k$.*

Example 4.1 (Practice:1.7 Text 21)**.** *Show there is a number c in the given interval such that $f(c) = 0$.*

$f(x) = e^x - 6 + 5x$, $(0, 1)$

Remark 4.1. *A useful result of the Intermediate Value Theorem is that a function may change signs only where it equals zero or at discontinuities. This is why the "sign chart" method of solving inequalities taught in algebra works. We will use this method extensively later.*

Example 4.2 (precalculus problem)**.** *Determine the solution to the inequality:*

$$\frac{-3x^2(x-1)}{(x-2)} \geq 0$$

5. Evaluating limits

Example 5.1. *Sketch the graph of* $y = \dfrac{x^2+3x-4}{(x+4)(x+1)^2}$ *including asymptotes.*

Example 5.2. *Determine the limit:* $\displaystyle\lim_{x\to 0} \frac{x^2+3x-4}{(x+4)(x+1)^2}$.

Example 5.3. *Determine the limit:* $\displaystyle\lim_{x\to -1} \frac{x^2+3x-4}{(x+4)(x+1)^2}$.

Example 5.4 (Practice: 1.7 Text Problems 23 and 67)**.** *Determine the limit:* $\displaystyle\lim_{x\to -4} \frac{x^2+3x-4}{(x+4)(x+1)^2}$.

Theorem 5.1. *If $f(x) = g(x)$ for all x in an open interval containing a (except possibly at a), then* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

FACT: To start a limit exercise in the form $\lim_{x \to a} f(x)$ where we have an algebraic expression for $f(x)$ involving the usual operations (plus, minus, exponents, logs, trig functions), begin by plugging in a to see what form you get. Depending on the result, we will take different approaches.

(1) $f(a)$ is a real number.

(2) $f(a)$ results in $\frac{c}{0}$, where $c \neq 0$.

(3) $f(a)$ results in $\frac{0}{0}$, $\pm\frac{\infty}{\infty}$, or $\infty - \infty$.

Example 5.5 (1.7 WP Homework Question 4, Text Problem 28)**.** *Use algebra to evaluate the limit:* $\lim_{h \to 0} \dfrac{(13+h)^2 - 13^2}{h}$.