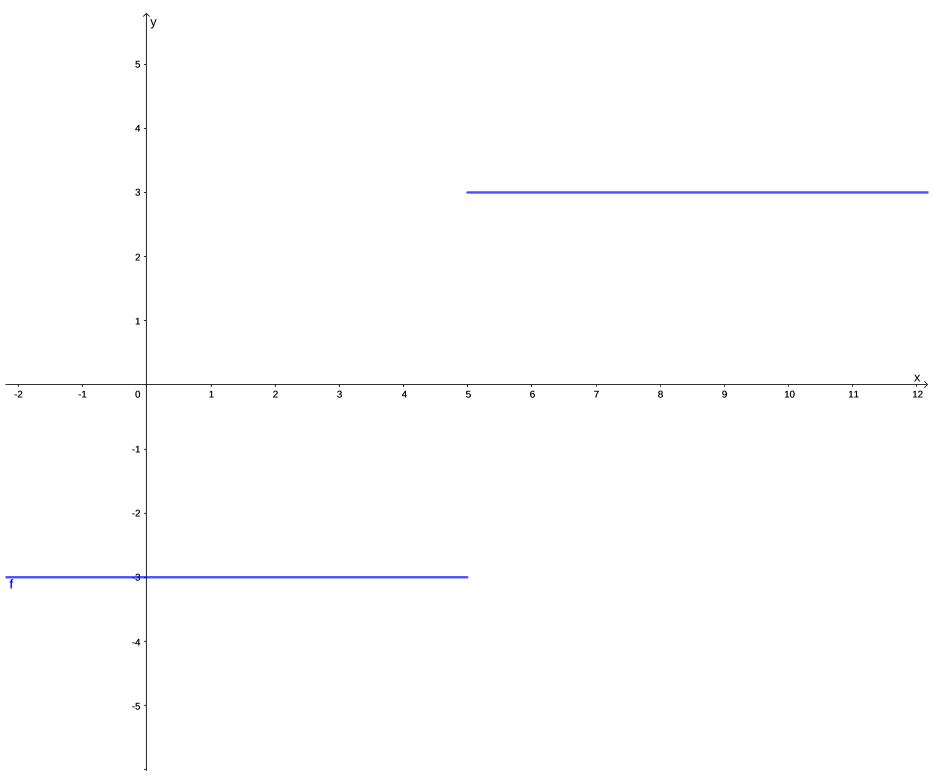
1. Chapter 1 Section 8: One-Sided Limits

We begin with an example of a function that is undefined at $x = 5$, but otherwise behave nicely. You are provided both a table of values and the graph of this function.

Example 1.1. *Consider the function* $f(x) = \dfrac{3x - 15}{\sqrt{x^2 - 10x + 25}}$ *and find* $\lim_{x \to 5} f(x)$.

x	$\frac{3x-5}{\sqrt{x^2-10x+25}}$	x	$\frac{3x-5}{\sqrt{x^2-10x+25}}$
5.5	3	4.5	-3
5.1	3	4.9	-3
5.01	3	4.99	-3
5.001	3	4.999	-3
5.0001	3	4.9999	-3



2. Left and Right Limits

Definition 2.1. *The* **limit of** $f(x)$**, as** x **approaches** a **from the left, equals** L *means that as x gets arbitrarily close to the value a AND $x < a$, the value of $f(x)$ gets close to the value L. This is also written*

$$\lim_{x \to a^-} f(x) = L$$

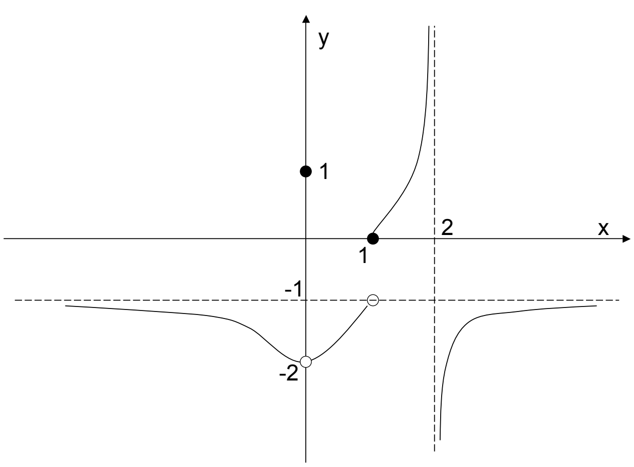
Definition 2.2. *The* **limit of** $f(x)$**, as** x **approaches** a **from the right, equals** L *means that as x gets arbitrarily close to the value a AND $x > a$, the value of $f(x)$ gets close to the value L. This is also written*

$$\lim_{x \to a^+} f(x) = L$$

Theorem 2.1. $\lim_{x \to a} f(x) = L$ *if and only if*

Example 2.1. *Go back to example 1.1 and find* $\lim_{x \to 5^-} f(x)$ *and* $\lim_{x \to 5^+} f(x)$*.*

Example 2.2 (1.8 WP Homework Question 1, Text 4)**.** *The graph of $y = f(x)$ is*



(1) $f(0) =$

(2) $\lim_{x \to 0^+} f(x) =$

(3) $\lim_{x \to 0^-} f(x) =$

(4) $\lim_{x \to 0} f(x) =$

(5) $f(1) =$

(6) $\lim_{x \to 1^+} f(x) =$

(7) $\lim_{x \to 1^-} f(x) =$

(8) $\lim_{x \to 1} f(x) =$

3. Infinite Limits

Definition 3.1. *The* **limit of** $f(x)$**, as** x **approaches** a**, is infinite** *means that as* x *gets arbitrarily close to the value* a*, the value of* $f(x)$ *gets arbitrarily large. This is also written* $\lim_{x\to a} f(x) = \infty$

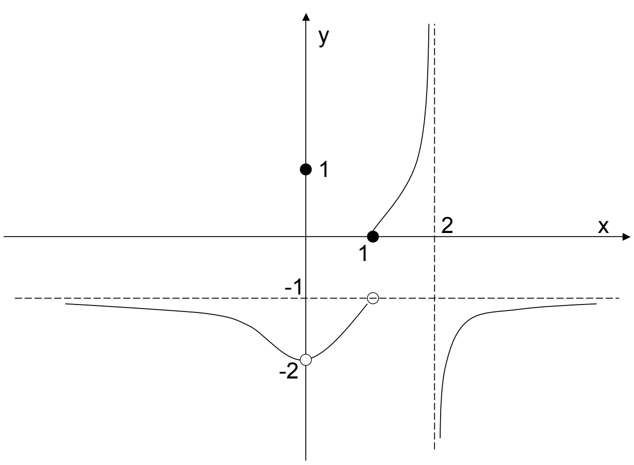
If the value of $|f(x)|$ *gets arbitrarily large, but* $f(x) < 0$*, for* x *close to* a*, then we write* $\lim_{x\to a} f(x) = -\infty$

Definition 3.2. *If* $\lim_{x\to a} |f(x)| = \infty$*,* $\lim_{x\to a^-} |f(x)| = \infty$*, or* $\lim_{x\to a^+} |f(x)| = \infty$*, then the vertical line* $x = a$ *is a* **vertical asymptote** *of the curve* $y = f(x)$*.*

Definition 3.3. *We say the limit as* x *approaches infinity is* L*, written* $\lim_{x\to\infty} f(x) = L$*, if for some* x *large enough the graph of* $y = f(x)$ *moves closer and closer to the line* $y = L$ *as one moves to the right. Moreover, in this case the graph* $y = f(x)$ *has a horizontal asymptote* $y = L$*.*

Definition 3.4. *We say the limit as* x *approaches negative infinity is* L*, written* $\lim_{x\to-\infty} f(x) = L$*, if for some* x *far enough to the left the graph of* $y = f(x)$ *moves closer and closer to the line* $y = L$ *as one moves further to the left. Moreover, in this case the graph* $y = f(x)$ *has a horizontal asymptote* $y = L$*.*

Example 3.1 (1.8 WP Homework Question 1, Text 4)**.** *The graph of* $y = f(x)$ *is*



(1) $\lim_{x\to 2^+} f(x) =$

(2) $\lim_{x\to 2^-} f(x) =$

(3) $\lim_{x\to\infty} f(x) =$

(4) $\lim_{x\to-\infty} f(x) =$

4. Quick Limits

The following limits are ones that you should know as quick facts that can be determined very quickly because you know the graphs. Review the graphs using the Desmos graph: https://www.desmos.com/calculator/alj9vflemw. Find these limits and then memorize them (or remember the graphs).

Given p is a positive integer constant and r is a positive real number.

(1) $\lim\limits_{x\to\infty} x^r =$

(2) $\lim\limits_{x\to-\infty} x^p =$

(3) $\lim\limits_{x\to\infty} \dfrac{1}{x^r} =$

(4) $\lim\limits_{x\to-\infty} \dfrac{1}{x^p} =$

(5) $\lim\limits_{x\to0^+} \dfrac{1}{x^p} =$

(6) $\lim\limits_{x\to0^-} \dfrac{1}{x^p} =$

(7) $\lim\limits_{x\to\infty} \sin x =$

(8) $\lim\limits_{x\to\infty} \cos x =$

(9) $\lim\limits_{x\to\infty} \arctan x =$

(10) $\lim\limits_{x\to-\infty} \arctan x =$

(11) $\lim\limits_{x\to\infty} e^x =$

(12) $\lim\limits_{x\to-\infty} e^x =$

(13) $\lim\limits_{x\to\infty} \ln x =$

(14) $\lim\limits_{x\to0^+} \ln x =$

(15) If f is a polynomial, $\lim\limits_{x\to\pm\infty} f(x) =$

5. Limit Laws

Find the formulas in Section 1.8 of the text to fill in the blank spaces in the formulas below in Theorem 5.1.

Theorem 5.1 (Limit Laws)**.** *If b, k, and n are constants and all limits involved exist (are real numbers), then*

(1) $\lim_{x\to a}[bf(x)] =$ ____________

(2) $\lim_{x\to a}[f(x)+g(x)] =$ ____________

(3) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(4) $\lim_{x\to a}[f(x)g(x)] =$ ____________

(5) $\lim_{x\to a}[f(x)/g(x)] =$ ____________

(6) $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$

(7) $\lim_{x\to a} k =$ ____________

(8) $\lim_{x\to a} x =$ ____________

(9) *If f is a function that you know from previous classes is "continuous" at a,* $\lim_{x\to a} f(x) = f(a)$.

(10) *If $f(x) = g(x)$ for all x in an open interval containing a (except possibly at a), then* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Example 5.1 (1.8 WP Homework Questions 2-5; Text 6, 8, 15)**.** *Given that* $\lim_{x\to c} f(x) = 4$, $\lim_{x\to c} g(x) = 6$, *and* $\lim_{x\to c} h(x) = -2$, *find the limit if it exists.*

(1) $\lim_{x\to c}(f(x)+5g(x)) =$

(2) $\lim_{x\to c}\sqrt{f(x)} =$

(3) $\lim_{x\to c}\dfrac{f(x)\cdot h(x)}{g(x)} =$

6. Absolute Values

Theorem 6.1. $\lim_{x \to a} f(x) = L$ *if and only if*

Example 6.1 (1.8 WP Homework Question 9, Text 42)**.** *Evaluate:* $\lim_{x \to -2^-} \frac{|x+2|}{x+2}$, $\lim_{x \to -2^+} \frac{|x+2|}{x+2}$, *and* $\lim_{x \to -2} \frac{|x+2|}{x+2}$

7. Piecewise Functions

Example 7.1 (1.8 WP Homework Question 10, Text 44)**.** *Let g be given by*

$$g(x) = \begin{cases} 2x+2 & \text{if } x < -2 \\ 1 & \text{if } -2 \leq x \leq 1 \\ 2 - x^2 & \text{if } x > 1 \end{cases}$$

(1) $\lim_{x \to -2^-} g(x) =$

(2) $\lim_{x \to -2^+} g(x) =$

(3) $\lim_{x \to -2} g(x) =$

(4) $g(-2) =$

Example 7.2 (1.8 WP Homework Question 11; Text 55, 56)**.** *Find the limit:* $\lim_{x\to\infty} \dfrac{4+x^2}{3+5x}$.

Example 7.3 (Practice: 1.8 Text 16-21)**.** *Sketch a graph of a function satisfying the following conditions:*

The domain for f *is all real numbers except* 2*,* f *is continuous on its domain,* $\lim_{x\to 2} f(x) = \infty$*,* $\lim_{x\to\infty} f(x) = -\infty$*,* $\lim_{x\to-\infty} f(x) = 1$*, and* $f(-2) = 0$*.*