1. Chapter 2 Section 5: The Second Derivative

Suppose we have a function $y = f(x)$ and we take the derivative of this function to get the function $f'(x)$. The derivative $f'(x)$ is a function, so we could now start over with this function to take the derivative again. The function we get by taking a derivative of a derivative is called the **second derivative** and one way to write the second derivative of f is $f''(x)$. f is called **twice differentiable** at a if $f''(a)$ exists.

Using Leibniz notation,

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Notice the derivative could be repeated any number of times and we write the n-th derivative of $y = f(x)$ as $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$. For now, we will concentrate on the second derivative.

Example 1.1. *Recall in the 2.3 course notes, Example 2.1: Let* $f(x) = \dfrac{1}{x+3}$*. Find* $f'(x)$ *using the definition of the derivative (the difference quotient). Use the answer for this example, to find* $f''(x)$*.*

Solution 1.1.1. *From Example 2.1 in Section 2.3 we found*

$$f'(x) = \frac{-1}{(x+3)^2}$$

We apply the difference quotient to this expression.

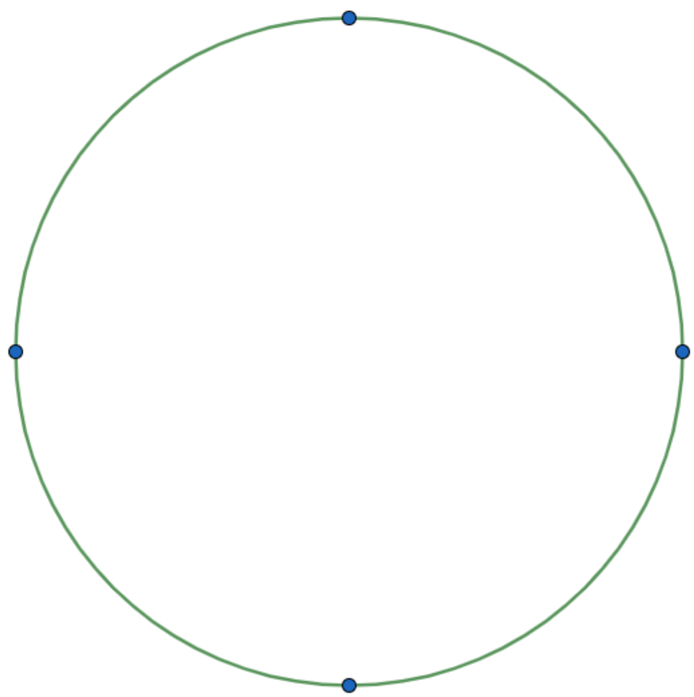
$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{-1}{(x+h+3)^2} - \frac{-1}{(x+3)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-(x+3)^2+(x+h+3)^2}{(x+h+3)^2(x+3)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-x^2-6x-9+x^2+h^2+2xh+6x+6h+9}{(x+h+3)^2(x+3)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{h^2+2xh+6h}{(x+h+3)^2(x+3)^2}}{h/1} \\
&= \lim_{h\to 0} \frac{h(h+2x+6)}{(x+h+3)^2(x+3)^2} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{(h+2x+6)}{(x+h+3)^2(x+3)^2} \\
&= \frac{(2x+6)}{(x+3)^2(x+3)^2} \\
&= \frac{(2(x+3))}{(x+3)^4} \\
&= \frac{2}{(x+3)^3}
\end{aligned}$$

So $f''(x) = \dfrac{2}{(x+3)^3}$

2. Concavity

Definitions 2.1. *Let f be a continuous function on the interval (a, b) and assume f is twice differentiable everywhere in the interval (a, b).*

(1) f is **concave up** *on the interval (a, b) if the graph of $y = f(x)$ lies above the line tangent to the graph at any point in the interval. Visually, a concave up shape is like the shape of a bowl that holds water.*
(2) f is **concave down** *on the interval (a, b) if the graph of $y = f(x)$ lies below the line tangent to the graph at any point in the interval. Visually, a concave down shape is like the shape of a bowl that is upside down.*
(3) If f changes concavity at $x = c$, then we say $(c, f(c))$ is an **inflection point***.*

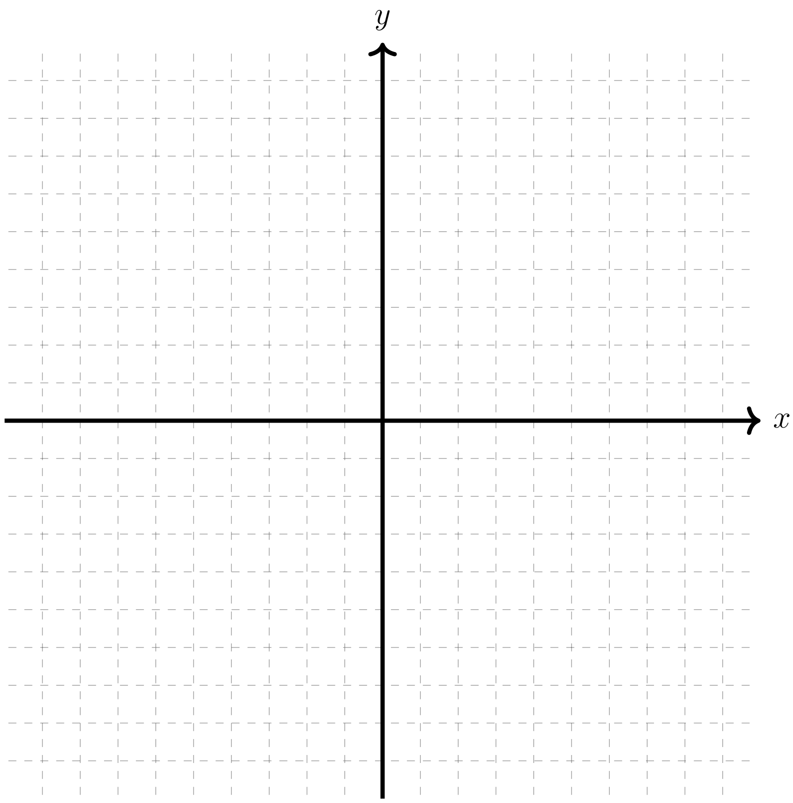


Theorem 2.1. *Suppose f is a function that is twice differentiable on the interval (a, b).*

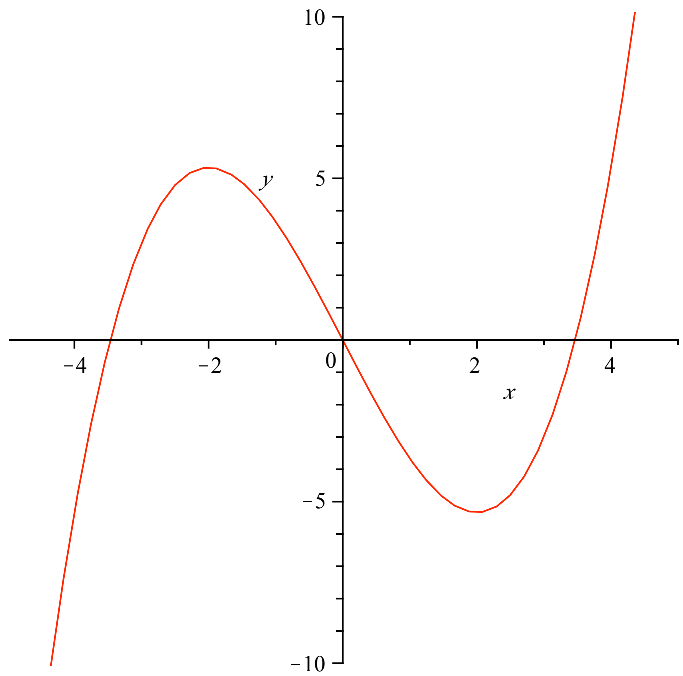
1. *$f''(x) = 0$ for all x in the interval (a, b) if and only if f is a linear function on (a, b).*
2. *$f''(x) > 0$ for all x in the interval (a, b), except possibly a finite number of points, if and only if f is concave up on (a, b).*
3. *$f''(x) < 0$ for all x in the interval (a, b), except possibly a finite number of points, if and only if f is concave down on (a, b).*

Example 2.1 (2.5 Text 45-46)**.** *Sketch a graph of a function satisfying all of the following conditions.*

f *is continuous on the domain* $(-\infty, 1) \cup (1, \infty)$, $\lim_{x \to \pm\infty} f(x) = 3$, $\lim_{x \to 1} f(x) = \infty$, $f(-4) = f(0) = 0$, $f(-2) = -2$, $f(4) = 6$, $f'(-2) = 0$, $f'(x) > 0$ *on* $(-2, 1)$, $f'(x) < 0$ *on* $(-\infty, -2) \cup (1, \infty)$, $f''(-4) = 0$, $f''(x) > 0$ *on* $(-4, 1) \cup (1, \infty)$, *and* $f''(x) < 0$ *on* $(-\infty, -4)$.



Example 2.2 (2.5 WP Homework Question 4, Text 22)**.** *Suppose the graph below is the graph of* $y = f(x)$*. From this graph determine the graphs of* $y = f'(x)$ *and* $y = f''(x)$*.*



Example 2.3 (2.5 Text 29)**.** *The Arctic Sea ice extent, the area of the sea covered by ice, grows seasonally over the winter months each year, typically from October to March, and is modeled by $G(t)$, in millions of square kilometers, t months after October 1, 2018.*

(1) *What is the sign of $G(t)$ for $0 < t < 4$?*

(2) *Suppose $G''(t) < 0$ for $0 < t < 4$. What does this tell us about how the Arctic Sea ice extent grows?*

(3) *Sketch a graph of $G(t)$ for $0 \leq t \leq 4$, given that $G(0) = 5.054$, $G(4) = 14.114$, and $G''(t)$ is as in part 2*

3. Motion

Suppose a particle moves along a straight line with position at time t given by $y = s(t)$.

Remarks 3.1.

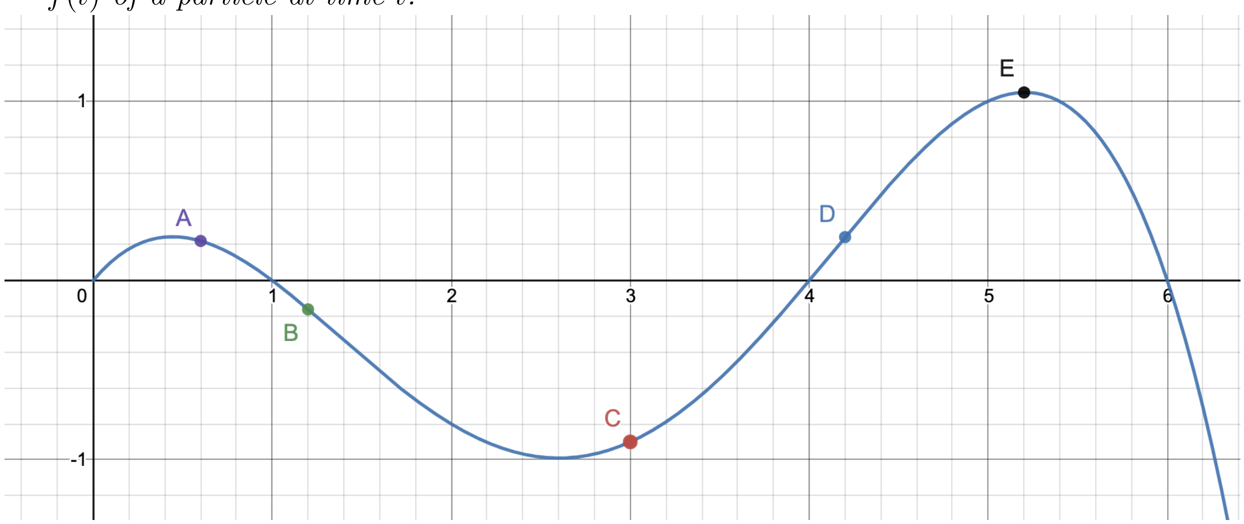
(1) *The (instantaneous)* **velocity** *at time t of the particle is* ____________

Recall: the sign of the velocity indicates which direction the particle is moving along the line. If the particle is changing direction at a given time, then the velocity must be zero at that time. We also say a particle is "at rest" when its velocity is 0, even if only for an instant.

(2) *The (instantaneous)* **acceleration** *at time t of the particle is* ____________

The sign of the acceleration indicates the direction of the force of acceleration. For example, the force of gravity is normally considered to be a negative acceleration since the direction of the force is down.

Example 3.1. *Refer to the given graph. Suppose this graph represents the position, $f(t)$ of a particle at time t.*



(1) *At which of the marked values is position positive?*

(2) *At which of the marked values is velocity positive?*

(3) *At which of the marked values is acceleration positive?*

(4) *At which of the marked values is position increasing?*

(5) *At which of the marked values is velocity increasing?*