1. Chapter 3 Section 6: Inverse Functions

Recall from pre-calculus the following definition.

Definition 1.1. *Let f be a one-to-one function. Then there is a function denoted f^{-1} called the* **inverse** *of f such that the domain and ranges of f and f^{-1} are reversed and $f(a) = b$ if and only if $f^{-1}(b) = a$.*

Remark 1.1. *Even when a function is not one-to-one, we can usually restrict the domain of a function and define the inverse on the restricted domain. Inverse trigonometric functions is an example of this.*

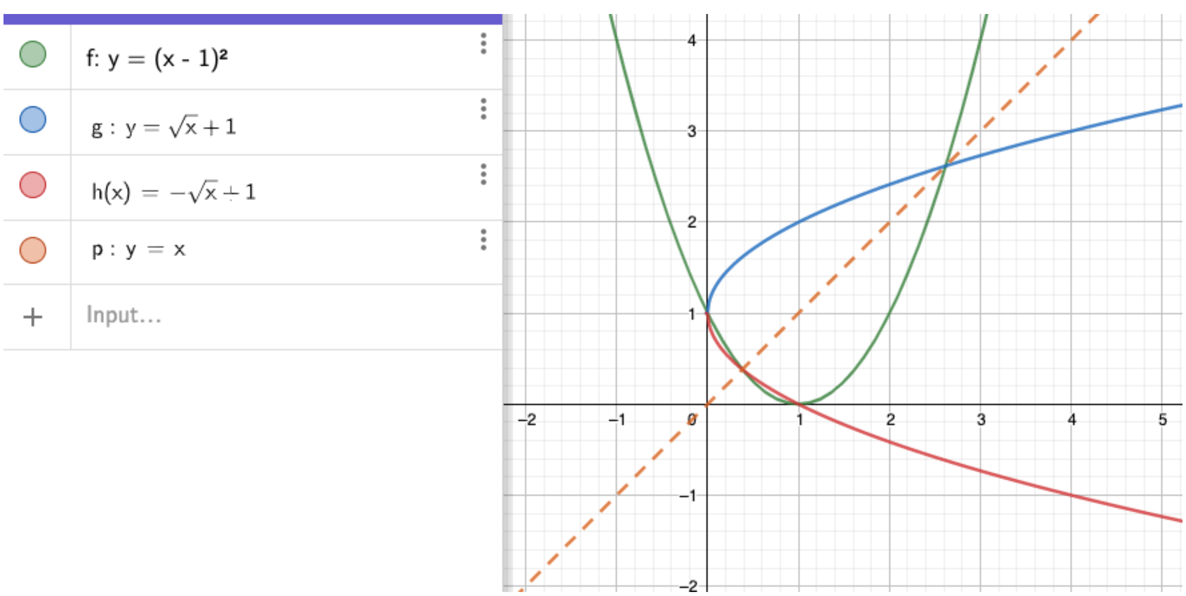
2. Properties of Inverses

(1) A function g is the inverse of f (and visa versa) if and only if $(f \circ g)(x) = x$ *and* $(g \circ f)(x) = x$.

(2) The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

(3) (a, b) is a point on the graph of $y = f(x)$ if and only if (b, a) is a point on the graph of $y = f^{-1}(x)$.

(4) The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



3. The Derivative of $f^{-1}(x)$

Given an invertible function $f(x)$, the derivative of $f^{-1}(x)$ is

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. From the definition of the inverse function, we know $f(f^{-1}(x)) = x$. So

$$\frac{d}{dx}[f(f^{-1}(x))] = \frac{d}{dx}[x].$$

Use the chain rule on the left side to get

$$f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) = 1.$$

Now solve for $\frac{d}{dx}f^{-1}(x)$:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

□

Example 3.1. *Find the derivative of $f^{-1}(x)$ when $f(x) = x^3$ two ways. First find the inverse function and take the derivative of the inverse function. Second, use the above formula for the derivative of the inverse.*

Example 3.2. *Suppose f is a differentiable function with a differentiable inverse. What is $(f^{-1})'(2)$ if $f(2) = 4$, $f(-3) = 2$, $f'(2) = 1$, $f'(0) = 2$, and $f'(-3) = 5$?*

Example 3.3. *Find the derivative of* $y = \ln x$ *using the derivative rule for an inverse and the derivative rule for the exponential function.*

(1) $\dfrac{d}{dx}[\ln(x)] =$

(2) $\dfrac{d}{dx}[\log_b(x)] =$

Example 3.4. *Find the derivative of* $y = (\ln x)^2 + \ln x^2 + \log_2 x$ *using the rules above.*

Rules Continued:

(3) $\frac{d}{dx}[\arctan(x)] =$

(4) $\frac{d}{dx}[\arcsin(x)] =$

Example 3.5. *Find the derivative of* $y = \arctan\sqrt{x^2+9}$ *using the rules above.*