1. Chapter 4 Section 2: Optimization

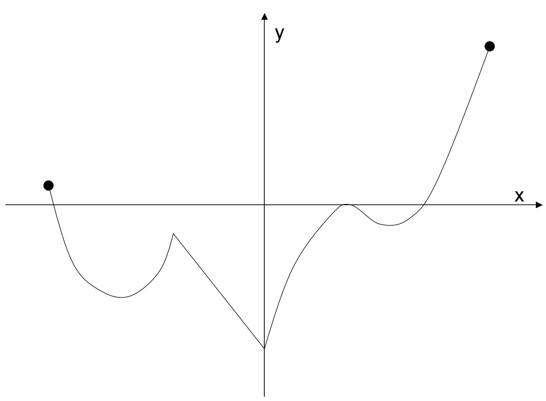
Definitions 1.1. *In the definitions below, assume $y = f(x)$ is a function with domain D, and c is a number in the domain of f.*

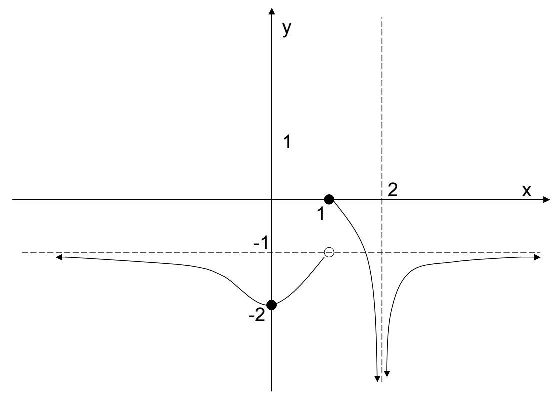
(1) *f has an* **absolute maximum** *or* **global maximum** *at $x = c$ if $f(c) \geq f(x)$ for all x in D. $f(c)$ is the* **(absolute) maximum (value)**.

(2) *f has an* **absolute minimum** *or* **global minimum** *at $x = c$ if $f(c) \leq f(x)$ for all x in D. $f(c)$ is the* **(absolute) minimum (value)**.

(3) *The minimum and maximum values are called the* **(absolute or global) extreme values** *of f.*

Example 1.1. *Find all global extrema and where they occur on each graph.*





Example 1.2. *Sketch the graph of a function that is continuous on $[-4, 4]$ with an absolute minimum at -4, an absolute maximum at 2, and local maximums at -1 and 2.*

2. Useful Theorems

The Extreme Value Theorem: If f is continuous on the closed interval $[a, b]$, then f will attain both a minimum and a maximum in the interval.

In other words, if you consider the interval $[a, b]$ as the domain of f, there will be at least one number c in $[a, b]$ where $f(c)$ is the absolute maximum value, and at least one number d in $[a, b]$ where $f(d)$ is the absolute minimum value.

Fermat's Theorem: If f has a local extreme at c, then $f'(c) = 0$ or $f'(c)$ is undefined.

This tells us that the only possible places where f may have a local extreme is where f has a **critical number**. Note that a critical number does not have to be a local extreme, but a local extreme has to be a critical number.

3. Global Maxima and Minima on a Closed Interval: Test the Candidates

To find the *absolute* minimum and maximum values of a *continuous* function f on a *closed interval* $[a, b]$:

Step 1. Find the critical numbers of f in (a, b).

Step 2. Evaluate f at all critical value(s) found in step 1 and at the end points, a and b.

Step 3. The largest value from step 3 is the maximum value, and the smallest value from step 3 is the minimum value.

Example 3.1. *Find the absolute maximum and absolute minimum values of f on the given interval.* $f(x) = e^{x^3+3x^2-9x}$, $[0, 3]$

Example 3.2. *Find the absolute maximum and absolute minimum values of f on the given interval.* $f(t) = \dfrac{t}{1+t^2}$, $[-1/2, 3]$

Example 3.3 (4.2 WP Homework Question 6, Text 18)**.** *Find the absolute maximum and absolute minimum values of f.* $f(t) = \dfrac{t}{1+t^2}$

Global Maxima and Minima on an Open Interval or on All Real Numbers

To find the *absolute* minimum and maximum values of a *continuous* function f on an *open interval* (a, b) (possibly $(-\infty, \infty)$):

Step 1. Find the critical numbers of f in (a, b).

Step 2. Sometimes the first derivative test will indicate if there is an absolute extreme or not and if it does then the rest of the steps are not necessary. However, the previous example cannot be solved just by applying the first derivative test.

Step 3. Evaluate f at all critical value(s) found in step 1.

Step 4. Find the limits of the function as x approaches the end points.

Step 5. Compare all the values in steps 3 and 4.
- (a) If the smallest value is a value in step 3 then that is the global minimum value.
- (b) If the smallest value is a value from step 4, then there is no global minimum on (a, b).
- (c) If the largest value is a value in step 3 then that is the global maximum value.
- (d) If the largest value is a value from step 4, then there is no global maximum on (a, b).

Exercise 3.1 (4.2 Text 47)**.** *For positive constants A and B, the force between two atoms in a molecule is given by*

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3}$$

where $r > 0$ is the distance between the two atoms. What value of r minimizes the force between the atoms?