1. Chapter 4 Section 8: Parametric Equations

Definition 1.1. *A set of equations that are defined using a single independent variable are called* **parametric equations**. *Often, we use t, called the* **parameter**, *as the independent variable to define the functions $x(t)$ and $y(t)$ (and perhaps $z(t)$). These equations define a* **parametric curve**, *C, in the plane (or in 3-space if $z(t)$ is given) such that the points on C are the set of points given by $(x(t), y(t))$.*

Example 1.1. *Sketch the parametric curve. Include an arrow to indicate the direction of the curve as t increases. Eliminate the parameter to find a Cartesian equation of the curve.* $x = \cos t$, $y = \sin t$

Remarks 1.1.

(1) *It is convenient to think of parametric equations as representing the position, (x, y), of a particle at time t. This is NOT the only application, just one that is useful to keep in mind.*

(2) *All functions can be represented using parametric equations. However, not all parametric equations define curves that can be represented using functions.*

(3) *A Cartesian equation of a curve can always be rewritten parametrically.*

(4) *One curve may be represented using many different parameterizations.*

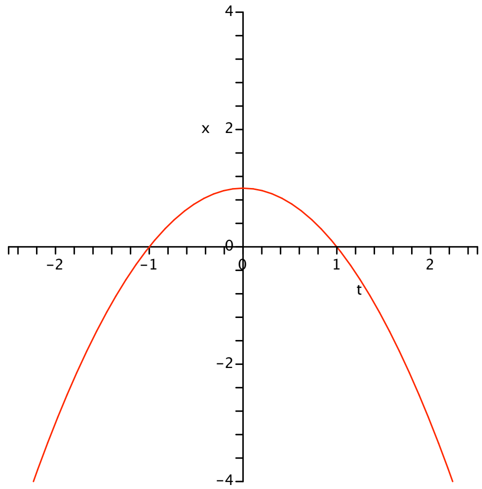
(5) *Finding a Cartesian equation of a curve from a parametrically defined curve can be extremely difficult and not necessarily desirable, but there are certain types that you should get comfortable with converting. The previous example is one you should learn.*

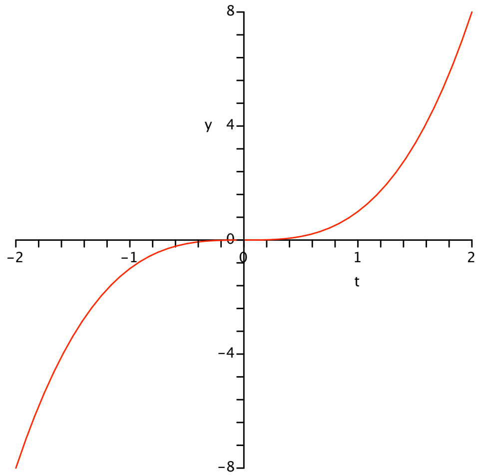
Example 1.2. *Let $x = x_0 + at$ and $y = y_0 + bt$, where x_0, y_0, a and b are all constants. Determine the shape of the graph in the xy-plane by eliminating the parameter.*

Example 1.3 (4.8 WP Homework Question 6, Text 39)**.** *A line is parameterized by $x = 2 + 3t$ and $y = 4 + 7t$.*

(1) What part of the line is obtained by restricting t to nonnegative numbers?
(2) What part of the line is obtained if t is restricted to $-1 \leq t \leq 0$?
(3) How should t be restricted to give the part of the line to the left of the y-axis?

Example 1.4. *Use the graphs of $x = f(t)$ (on the left) and $y = g(t)$ (on the right) to sketch the parametric curve $x = f(t), y = g(t)$. Include an arrow to indicate the direction of the curve as t increases.*





2. The Derivative and Slopes of Tangent Lines

Example 2.1. *Review Example 1.1: Sketch the parametric curve. Include an arrow to indicate the direction of the curve as t increases.* $x = \cos t$, $y = \sin t$

Sketch the graphs below and the tangent lines at $t = 0, \pi/4,$ and $\pi/2$.

(1) The graph with t as the horizontal axis and x as the vertical axis.

(2) The graph with t as the horizontal axis and y as the vertical axis.

(3) The graph with x as the horizontal axis and y as the vertical axis.

Notice the slope of the line tangent to the graph in 1 is found using ____________.

The slope of the line tangent to the graph in 2 is found using ____________.

To find the slope of the line tangent to the graph in 3 we need to find ...

Example 2.2. $x = t + \ln t$ *and* $y = t - \ln t$

(1) *Find* $\dfrac{dy}{dx}$.

(2) *Find the equation of the line tangent to graph at* $t = e$. *Note: Unless otherwise specified, the "the graph" is referring to the graph in the* xy*-plane.*

(3) *Find the intervals on which the graph is increasing and decreasing.*

3. The Second Derivative and Concavity

Recall concavity of a graph is found using ...

Example 3.1. $x = t + \ln t$ *and* $y = t - \ln t$

(1) *Find* $\frac{d^2y}{dx^2}$

(2) *Find the intervals on which the graph is concave up and concave down.*

(3) *Sketch the graph.*

Definitions 3.1. *Let $x = x(t)$ and $y = y(t)$ be parametric equations describing the position of an object at time t.*

1. *The quantity $v_x = \dfrac{dx}{dt}$ is the* **(instantaneous) velocity** *in the x-direction.*
2. *The quantity $v_y = \dfrac{dy}{dt}$ is the* **(instantaneous) velocity** *in the y-direction.*
3. *The* **velocity vector** *is written $\vec{v} = v_x\vec{i} + v_y\vec{j}$. Alternative notations for this vector are $\mathbf{v} = \vec{v} = \hat{v} = \langle v_x, v_y \rangle$.*
4. *The* **(instantaneous) speed** *of the object is the magnitude of the velocity vector:*

$$Speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Example 3.2. *If a projectile is fired with initial velocity v_0 meters per second at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations*

$$x = (v_0 \cos\alpha)t \qquad y = (v_o \sin\alpha)t - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity $(9.8m/s^2)$.

1. *If a gun is fired at $60°$ and $v_0 = 400\ m/s$, when will the bullet hit the ground? How far from the gun will it hit the ground?*
2. *Show the path is parabolic by eliminating the parameter.*
3. *Find the speed of the bullet when it hits the ground.*