CHAPTER 4

Applications of Methods of Proof

1. Set Operations

1.1. Set Operations. The set-theoretic operations, intersection, union, and
complementation, defined in Chapter 1.1 Introduction to Sets are analogous to the
operations $\land$, $\lor$, and $\neg$, respectively, that were defined for propositions. Indeed, each
set operation was defined in terms of the corresponding operator from logic. We will
discuss these operations in some detail in this section and learn methods to prove
some of their basic properties.

Recall that in any discussion about sets and set operations there must be a set,
called a universal set, that contains all others sets to be considered. This term is a
bit of a misnomer: logic prohibits the existence of a “set of all sets,” so that there
is no one set that is “universal” in this sense. Thus the choice of a universal set will
depend on the problem at hand, but even then it will in no way be unique. As a rule
we usually choose one that is minimal to suit our needs. For example, if a discussion
involves the sets \{1, 2, 3, 4\} and \{2, 4, 6, 8, 10\}, we could consider our universe to be
the set of natural numbers or the set of integers. On the other hand, we might be
able to restrict it to the set of numbers \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.

We now restate the operations of set theory using the formal language of logic.

1.2. Equality and Containment.

Definition 1.2.1. Sets $A$ and $B$ are equal, denoted $A = B$, if

$$\forall x[x \in A \leftrightarrow x \in B]$$

Note: This is equivalent to

$$\forall x[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$$.

Definition 1.2.2. Set $A$ is contained in set $B$ (or $A$ is a subset of $B$), denoted
$A \subseteq B$, if

$$\forall x[x \in A \rightarrow x \in B]$$.
The above note shows that
\[ A = B \]
iff
\[ A \subseteq B \text{ and } B \subseteq A. \]

1.3. Union and Intersection.

Definitions 1.3.1.

- The union of \( A \) and \( B \),
  \[ A \cup B = \{ x | (x \in A) \lor (x \in B) \}. \]
- The intersection of \( A \) and \( B \),
  \[ A \cap B = \{ x | (x \in A) \land (x \in B) \}. \]
- If \( A \cap B = \emptyset \), then \( A \) and \( B \) are said to be disjoint.

1.4. Complement.

Definition 1.4.1. The complement of \( A \)
\[ \overline{A} = \{ x \in U | \neg(x \in A) \} = \{ x \in U | x \not\in A \}. \]

Discussion

There are several common notations used for the complement of a set. For example, \( A^c \) is often used to denote the complement of \( A \). You may find it easier to type \( A^c \) than \( \overline{A} \), and you may use this notation in your homework.

1.5. Difference.

Definition 1.5.1. The difference of \( A \) and \( B \), or the complement of \( B \) relative to \( A \),
\[ A - B = A \cap B. \]

Definition 1.5.2. The symmetric difference of \( A \) and \( B \),
\[ A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B). \]

Discussion

The difference and symmetric difference of two sets are new operations, which were not defined in Module 1.1. Notice that \( B \) does not have to be a subset of \( A \) for
the difference to be defined. This gives us another way to represent the complement of a set \( A \); namely, \( \overline{A} = U - A \), where \( U \) is the universal set.

The definition of the difference of two sets \( A \) and \( B \) in some universal set, \( U \), is equivalent to \( A - B = \{ x \in U | (x \in A) \land \neg(x \in B) \} \).

Many authors use the notation \( A \setminus B \) for the difference \( A - B \).

The symmetric difference of two sets corresponds to the logical operation \( \oplus \), the exclusive "or".

The definition of the symmetric difference of two sets \( A \) and \( B \) in some universal set, \( U \), is equivalent to
\[
A \oplus B = \{ x \in U | [(x \in A) \land \neg(x \in B)] \lor [\neg(x \in A) \land (x \in B)] \}.
\]

**1.6. Product.**

**Definition 1.6.1.** The (Cartesian) Product of two sets, \( A \) and \( B \), is denoted \( A \times B \) and is defined by
\[
A \times B = \{(a, b) | a \in A \land b \in B \}
\]

**1.7. Power Set.**

**Definition 1.7.1.** Let \( S \) be a set. The power set of \( S \), denoted \( \mathcal{P}(S) \) is defined to be the set of all subsets of \( S \).

**Discussion**

Keep in mind the power set is a set where all the elements are actually sets and the power set should include the empty set and itself as one of its elements.

**1.8. Examples.**

**Example 1.8.1.** Assume: \( U = \{a, b, c, d, e, f, g, h\} \), \( A = \{a, b, c, d, e\} \), \( B = \{c, d, e, f\} \), and \( C = \{a, b, c, g, h\} \). Then

\( (a) \ A \cup B = \{a, b, c, d, e, f\} \)
\( (b) \ A \cap B = \{c, d, e\} \)
\( (c) \ \overline{A} = \{f, g, h\} \)
\( (d) \ \overline{B} = \{a, b, g, h\} \)
\( (e) \ A - B = \{a, b\} \)
\( (f) \ B - A = \{f\} \)
\( (g) \ A \oplus B = \{a, b, f\} \)
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(h) \((A \cup B) \cap C = \{a, b, c\}\)

(i) \(A \times B = \{(a, c), (a, d), (a, e), (a, f), (b, c), (b, d), (b, e), (b, f), (c, c), (c, d), (c, e),
\( (c, f), (d, c), (d, d), (d, e), (d, f), (e, c), (e, d), (e, e), (e, f)\}\)

(j) \(\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, \{a, b, c, d, e\}\}\)

(k) \(|\mathcal{P}(A)| = 32\)

Exercise 1.8.1. Use the sets given in Example 1.8.1 to find

\( (1) B \times A \)
\( (2) \mathcal{P}(B) \)
\( (3) |\mathcal{P}(U)| \)

Example 1.8.2. Let the universal set be \(U = \mathbb{Z}^+\) the set of all positive integers, let \(P\) be the set of all prime (positive) integers, and let \(E\) be the set of all positive even integers. Then

(a) \(P \cup E = \{n \in \mathbb{Z}^+|n \text{ is prime or even}\}\),
(b) \(P \cap E = \{2\}\),
(c) \(\overline{P}\) is the set of all positive composite integers,
(d) \(\overline{E}\) is the set of all positive odd integers, \(\{2n + 1|n \in \mathbb{N}\}\),
(e) \(P - E = \{2, 4, 8, 10, \ldots\} = \{2n|n \in \mathbb{Z}^+ \land n \geq 2\}\),
(f) \(E - P = \{4, 6, 8, 10, \ldots\} = \{2n|n \in \mathbb{Z}^+ \land n \geq 2\}\),
(g) \(E \oplus P = \{n \in \mathbb{Z}^+|(n \text{ is prime or even}) \land n \neq 2\}\)

Exercise 1.8.2.

(1) If \(|A| = n\) and \(|B| = m\), how many elements are in \(A \times B\)?

(2) If \(S\) is a set with \(|S| = n\), what is \(|\mathcal{P}(S)|\)?

Exercise 1.8.3. Does \(A \times B = B \times A\)? Prove your answer.

1.9. Venn Diagrams. A Venn Diagram is a useful geometric visualization tool when dealing with three or fewer sets. The Venn Diagram is generally set up as follows:

- The Universe \(U\) is the rectangular box.
- A set are represented by a circle and its interior.
- In the absence of specific knowledge about the relationships among the sets being represented, the most generic relationships should be depicted.

Discussion
Venn Diagrams can be very helpful in visualizing set operations when you are dealing with three or fewer sets (not including the universal set). They tend not to be as useful, however, when considering more than three sets. Although Venn diagrams may be helpful in visualizing sets and set operations, they will not be used for proving set theoretic identities.

1.10. Examples.

Example 1.10.1. The following Venn Diagrams illustrate generic relationships between two and three sets, respectively.

Example 1.10.2. This Venn Diagram represents the difference $A - B$ (the shaded region).

The figures in the examples above show the way you might draw the Venn diagram if you aren’t given any particular relations among the sets. On the other hand, if you knew, for example, that $A \subseteq B$, then you would draw the set $A$ inside of $B$. 
1.11. Set Identities.

Example 1.11.1. Prove that the complement of the union is the intersection of the complements:

\[ \overline{A \cup B} = \overline{A} \cap \overline{B}. \]

Proof 1. One way to show the two sets are equal is to use the fact that

\[ \overline{A \cup B} = \overline{A} \cap \overline{B} \]

iff

\[ \overline{A \cup B} \subseteq \overline{A} \cap \overline{B} \text{ and } \overline{A} \cap \overline{B} \subseteq \overline{A \cup B}. \]

Step 1. Show \( \overline{A \cup B} \subseteq \overline{A} \cap \overline{B} \).

Assume \( x \) is an arbitrary element of \( \overline{A \cup B} \) (and show \( x \in \overline{A} \cap \overline{B} \)). Since \( x \in \overline{A \cup B} \), \( x \notin A \cup B \). This means \( x \notin A \) and \( x \notin B \) (De Morgan’s Law). Hence \( x \in \overline{A \cap B} \). Thus, by Universal Generalization,

\[ \forall x \left[ x \in (A \cup B) \rightarrow x \in (\overline{A} \cap \overline{B}) \right] \]

so that, by definition,

\[ \overline{A \cup B} \subseteq \overline{A} \cap \overline{B}. \]

Step 2. Show \( \overline{A} \cap \overline{B} \subseteq \overline{A \cup B} \).

Suppose \( x \) is an arbitrary element of \( \overline{A} \cap \overline{B} \). Then \( x \notin A \) and \( x \notin B \). Therefore, \( x \notin A \cup B \) (De Morgan’s Law). This shows \( x \in \overline{A \cup B} \). Thus, by Universal Generalization,

\[ \forall x \left[ x \in (\overline{A} \cap \overline{B}) \rightarrow x \in (\overline{A \cup B}) \right] \]

so that, by definition,

\[ \overline{A} \cap \overline{B} \subseteq \overline{A \cup B}. \]

\[ \square \]

Proof 2. The following is a second proof of the same result, which emphasizes more clearly the role of the definitions and laws of logic. We will show

\[ \forall x \left[ x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B} \right]. \]
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<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
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</thead>
<tbody>
<tr>
<td>∀x: x ∈ (A ∪ B) ⇔ x (\notin [A ∪ B])</td>
<td>Definition of complement</td>
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<tr>
<td>⇔ ¬[x ∈ A ∪ B]</td>
<td>Definition of (\notin)</td>
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<td>⇔ ¬[(x ∈ A) \lor (x ∈ B)]</td>
<td>Definition of union</td>
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<td>⇔ ¬(x ∈ A) ∧ ¬(x ∈ B)</td>
<td>De Morgan’s Law</td>
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<tr>
<td>⇔ (x ∈ (\overline{A})) ∧ (x ∈ (\overline{B}))</td>
<td>Definition of complement</td>
</tr>
<tr>
<td>⇔ x ∈ (\overline{A} \cap \overline{B})</td>
<td>Definition of intersection</td>
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</tbody>
</table>

Hence ∀x[x ∈ \(A ∪ B\) ↔ x ∈ \(\overline{A} \cap \overline{B}\)] is a tautology. □

(In practice we usually omit the formality of writing ∀x in the initial line of the proof and assume that x is an arbitrary element of the universe of discourse.)

**Proof 3.** A third way to prove this identity is to build a **membership table** for the sets \(A ∪ B\) and \(A \cap B\), and show the membership relations for the two sets are the same. The 1’s represent membership in a set and the 0’s represent nonmembership.

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<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>(A ∪ B)</th>
<th>(\overline{A} ∪ \overline{B})</th>
<th>(A)</th>
<th>(B)</th>
<th>(\overline{A} \cap \overline{B})</th>
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Compare this table to the truth table for the proof of De Morgan’s Law:

\(\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)\)

□

**Discussion**

A set identity is an equation involving sets and set operations that is true for all possible choices of sets represented by symbols in the identity. These are analogous to identities such as

\[(a + b)(a - b) = a^2 - b^2\]
that you encounter in an elementary algebra course.

There are various ways to prove an identity, and three methods are covered here. This is a good place to be reminded that when you are proving an identity, you must show that it holds in all possible cases. Remember, giving an example does not prove an identity. On the other hand, if you are trying to show that an expression is not an identity, then you need only provide one counterexample. (Recall the negation of $\forall x P(x)$ is $\exists x \neg P(x)$).

Proof 1 establishes equality by showing each set is a subset of the other. This method can be used in just about any situation.

Notice that in Proof 1 we start with the assumption, $x \in A \cup B$, where $x$ is otherwise an arbitrary element in some universal set. If we can show that $x$ must then be in $A \cap B$, then we will have established

$$\forall x, [x \in A \cup B] \to [x \in A \cap B].$$

That is, the *modus operandi* is to prove the implications hold for an arbitrary element $x$ of the universe, concluding, by Universal Generalization, that the implications hold for all such $x$.

Notice the way De Morgan’s Laws are used here. For example, in the first part of Proof 1, we are given that $x \notin (A \cup B)$. This means

$$\neg [x \in (A \cup B)] \iff \neg [(x \in A) \lor (x \in B)] \iff [(x \notin A) \land (x \notin B)].$$

Proof 2 more clearly exposes the role of De Morgan’s Laws. Here we prove the identity by using propositional equivalences in conjunction with Universal Generalization. When using this method, as well as any other, you must be careful to provide reasons.

Proof 3 provides a nice alternative when the identity only involves a small number of sets. Here we show two sets are equal by building a member table for the sets. The member table has a 1 to represent the case in which an element is a member of the set and a 0 to represent the case when it is not. The set operations correspond to a logical connective and one can build up to the column for the set desired.

You will have proved equality if you demonstrate that the two columns for the sets in question have the exact same entries. Notice that all possible membership relations of an element in the universal set for the sets $A$ and $B$ are listed in the first two columns of the membership table. For example, if an element is in both $A$ and $B$ in our example, then it satisfies the conditions in the first row of the table. Such an element ends up in neither of the two sets $A \cup B$ nor $A \cap B$.

This is very straightforward method to use for proving a set identity. It may also be used to prove containment. If you are only trying to show the containment $M \subseteq N$, you would build the membership table for $M$ and $N$ as above. Then you would look in every row where $M$ has a 1 to see that
N also has a 1. However, you will see examples in later modules where a membership table cannot be created. It is not always possible to represent all the different possibilities with a membership table.

Example 1.11.2. Prove the identity \((A \cup B) \cap C = (A \cap C) \cup (B \cap C)\)

Proof 1. Suppose \(x\) is an arbitrary element of the universe.

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<tr>
<th>Assertion</th>
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<td>(x \in (A \cup B) \cap C)</td>
<td>definition of intersection</td>
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<td>(\Leftrightarrow [x \in (A \cup B)] \land [x \in C])</td>
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<td>(\Leftrightarrow [(x \in A) \lor (x \in B)] \land [x \in C])</td>
<td>definition of union</td>
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<tr>
<td>(\Leftrightarrow [(x \in A) \land (x \in C)] \lor [(x \in B) \land (x \in C)])</td>
<td>distributive law of “and” over “or”</td>
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<tr>
<td>(\Leftrightarrow [x \in (A \cap C)] \lor [(x \in (B \cap C)])</td>
<td>definition of intersection</td>
</tr>
<tr>
<td>(\Leftrightarrow x \in [(A \cap C) \cup (B \cap C))</td>
<td>definition of union</td>
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Proof 2. Build a membership table:

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Since the columns corresponding to \((A \cup B) \cap C\) and \((A \cap C) \cup (B \cap C)\) are identical, the two sets are equal.
Example 1.11.3. Prove \((A - B) - C \subseteq A - (B - C)\).

Proof. Consider the membership table:

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<th>(A - B)</th>
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Notice the only 1 in the column for \((A - B) - C\) is the fourth row. The entry in the same row in the column for \(A - (B - C)\) is also a 1, so \((A - B) - C \subseteq A - (B - C)\). \(\square\)

Exercise 1.11.1. Prove the identity \(A - B = A \cap \overline{B}\) using the method of Proof 2 in Example 1.11.1.

Exercise 1.11.2. Prove the identity \(A - B = A \cap \overline{B}\) using the method of Proof 3 in Example 1.11.1.

Exercise 1.11.3. Prove the identity \((A \cup B) - C = (A - C) \cup (B - C)\) using the method of Proof 1 in Example 1.11.1.

Exercise 1.11.4. Prove the identity \((A \cup B) - C = (A - C) \cup (B - C)\) using the method of Proof 2 in Example 1.11.1.

Exercise 1.11.5. Prove the identity \((A \cup B) - C = (A - C) \cup (B - C)\) using the method of Proof 3 in Example 1.11.1.

1.12. Union and Intersection of Indexed Collections.

Definition 1.12.1. The union and intersection of an indexed collection of sets

\[\{A_1, A_2, A_3, \ldots, A_n\}\]

can be written as

\[\bigcap_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n\]
and

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n, \]

respectively.

1.13. Infinite Unions and Intersections.

**Definition 1.13.1.** The union and intersection of an indexed collection of infinitely many sets

\[ \{A_1, A_2, A_3, \ldots, \} \]

can be written as

\[ \bigcup_{i=1}^{\infty} A_i = \{ a | a \in A_i \text{ for some } i \text{ in } \mathbb{Z}^+ \} \]

and

\[ \bigcap_{i=1}^{\infty} A_i = \{ a | a \in A_i \text{ for all } i \text{ in } \mathbb{Z}^+ \} \]

**Discussion**

If you have a collection of more than two sets, you can define the intersection and the union of the sets as above. (Since the operations are associative, it isn’t necessary to clutter the picture with parentheses.) The notation is similar to the \( \Sigma \) notation used for summations. The subscript is called an index and the collection of sets is said to be indexed by the set of indices. In the example, the collection of sets is \( \{A_1, A_2, \ldots, A_n\} \), and the set of indices is the set \( \{1, 2, \ldots, n\} \). There is no requirement that sets with different indices be different. In fact, they could all be the same set. This convention is very useful when each of the sets in the collection is naturally described in terms of the index (usually a number) it has been assigned.

An equivalent definition of the union and intersection of an indexed collection of sets is as follows:

\[ \bigcup_{i=1}^{n} A_i = \{ x | \exists i \in \{1, 2, \ldots, n\} \text{ such that } x \in A_i \} \]

and

\[ \bigcap_{i=1}^{n} A_i = \{ x | \forall i \in \{1, 2, \ldots, n\}, x \in A_i \}. \]
Another standard notation for unions over collections of indices is

\[ \bigcup_{i \in \mathbb{Z}^+} A_i = \bigcup_{i=1}^{\infty} A_i. \]

More generally, if \( J \) is any set of indices, we can define

\[ \bigcup_{i \in J} A_i = \{ x | \exists i \in J \text{ such that } x \in A_i \}. \]


**Example 1.14.1.** Let \( A_i = [i, i+1) \), where \( i \) is a positive integer. Then

- \( \bigcup_{i=1}^{n} A_i = [1, n+1) \), and
- \( \bigcap_{i=1}^{n} A_i = \emptyset \), if \( n > 1 \).
- \( \bigcup_{i=1}^{\infty} A_i = [1, \infty) \)
- \( \bigcap_{i=1}^{\infty} A_i = \emptyset \)

**Discussion**

This is an example of a collection of subsets of the real numbers that is naturally indexed. If \( A_i = [i, i+1) \), then \( A_1 = [1, 2) \), \( A_2 = [2, 3) \), \( A_3 = [3, 4) \), etc. It may help when dealing with an indexed collection of sets to explicitly write out a few of the sets as we have done here.

**Example 1.14.2.** Suppose \( C_i = \{i-2, i-1, i, i+1, i+2\} \), where \( i \) denotes an arbitrary natural number. Then

- \( C_0 = \{-2, -1, 0, 1, 2\} \),
- \( C_1 = \{-1, 0, 1, 2, 3\} \),
- \( C_2 = \{0, 1, 2, 3, 4\} \),
- \( \bigcup_{i=0}^{n} C_i = \{-2, -1, 0, 1, \ldots, n, n+1, n+2\} \)
- \( \bigcap_{i=0}^{4} C_i = \{2\} \)
1. SET OPERATIONS

- $\bigcap_{i=0}^{n} C_i = \emptyset$ if $n > 4$.
- $\bigcup_{i=0}^{\infty} C_i = \{-2, -1, 0, 1, 2, 3, \ldots\}$
- $\bigcap_{i=0}^{\infty} C_i = \emptyset$

**Exercise 1.14.1.** For each positive integer $k$, let $A_k = \{kn | n \in \mathbb{Z}\}$. For example,

- $A_1 = \{n | n \in \mathbb{Z}\} = \mathbb{Z}$
- $A_2 = \{2n | n \in \mathbb{Z}\} = \{\ldots, -2, 0, 2, 4, 6, \ldots\}$
- $A_3 = \{3n | n \in \mathbb{Z}\} = \{\ldots, -3, 0, 3, 6, 9, \ldots\}$

Find

1. $\bigcap_{k=1}^{10} A_k$

2. $\bigcap_{k=1}^{m} A_k$, where $m$ is an arbitrary positive integer.

**Exercise 1.14.2.** Use the definition for $A_k$ in exercise 1.14.1 to answer the following questions.

1. $\bigcap_{i=1}^{\infty} A_i$
2. $\bigcup_{i=1}^{\infty} A_i$

**1.15. Computer Representation of a Set.** Here is a method for storing subsets of a given, finite universal set:

Order the elements of the universal set and then assign a bit number to each subset $A$ as follows. A bit is 1 if the element corresponding to the position of the bit in the universal set is in $A$, and 0 otherwise.

**Example 1.15.1.** Suppose $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, with the obvious ordering. Then

- The bit string corresponding to $A = \{2, 4, 6, 8, 10\}$ is 0101010101.
- The bit string corresponding to $B = \{1, 2, 3\}$ is 1111000000.
Discussion

There are many ways sets may be represented and stored in a computer. One such method is presented here. Notice that this method depends not only on the universal set, but on the *order* of the universal set as well. If we rearrange the order of the universal set given in Example 1.15.1 to \( U = \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\} \), then the bit string corresponding to the subset \( \{1, 2, 3, 4\} \) would be 0000001111.

The beauty of this representation is that set operations on subsets of \( U \) can be carried out formally using the corresponding bitstring operations on the bit strings representing the individual sets.

**Example 1.15.2.** For the sets \( A, B \subseteq U \) in Example 1.15.1:

\[
A = \overline{0101010101} = 1010101010 = \{1, 3, 5, 7, 9\}
\]

\[
A \cup B = 0101010101 \lor 1111000000 = 1111010101 = \{1, 2, 3, 4, 6, 8, 10\}
\]

\[
A \cap B = 0101010101 \land 1111000000 = 0101000000 = \{2, 4\}
\]

\[
A \oplus B = 0101010101 \oplus 1111000000 = 1010010101 = \{1, 3, 6, 8, 10\}
\]

**Exercise 1.15.1.** Let \( U = \{a, b, c, d, e, f, g, h, i, j\} \) with the given alphabetical order. Let \( A = \{a, e, i\} \), \( B = \{a, b, d, e, g, h, j\} \), and \( C = \{a, c, e, g, i\} \).

1. Write out the bit string representations for \( A, B, \) and \( C \).
2. Use these representations to find
   (a) \( \overline{C} \)
(b) $A \cup B$
(c) $A \cap B \cap C$
(d) $B - C$