MAD 3105 PRACTICE TEST 2 SOLUTIONS

1. Let R be the relation defined below. Determine which properties, reflexive, irreflexive, symmetric, antisymmetric, transitive, the relation satisfies. Prove each answer.
   (a) R is the relation on a set of all people given by two people a and b are such that $(a, b) \in R$ if and only if a and b are enrolled in the same course at FSU.
      reflexive: Yes. Each person is in the same class with themselves.
      irreflexive: No. See the previous.
      symmetric: Yes, if a and b are enrolled in the same course, then b and a are enrolled in the same course.
      antisymmetric: No. Choose any two different people enrolled in this course. This provides a counterexample.
      transitive: No. Person a and b may be enrolled in one course, and person c may be enrolled in a course with b, but different from the first course. This situation provides a counterexample.
   (b) R is the relation on $\{a, b, c\}$, $R = \{(a, b), (b, a), (b, b), (c, c)\}$
      reflexive: No. (a, a) is not in R.
      irreflexive: No. (b, b) is in R.
      symmetric: Yes. For each pair $(x, y) \in R$ you can check that the pair $(y, x) \in R$.
      antisymmetric: No. $(a, b), (b, a) \in R$ and $a \neq b$.
      transitive: No. $(a, b), (b, a) \in R$, but $(a, a) \notin R$.
   (c) R is the relation on the set of positive integers given by mRn if and only if $gcd(m, n) > 1$.
      reflexive: No. $gcd(1, 1) = 1 \not> 1$, so $(1, 1) \notin R$.
      irreflexive: No. $gcd(2, 2) = 2 > 1$, so $(2, 2) \in R$.
      symmetric: Yes. $gcd(m, n) = gdc(n, m)$, so if $gcd(m, n) > 1$ then $gcd(n, m) > 1$.
      antisymmetric: No. $gcd(4, 2) = gcd(2, 4) = 2 > 1$ so $(2, 4), (4, 2) \in R$. But $2 \neq 4$.
      transitive: No. $gcd(2, 6) = 2 > 1$ and $gcd(6, 3) = 3 > 1$, so $(2, 6), (6, 3) \in R$. But $gcd(2, 3) = 1 \not> 1$ so $(2, 3) \notin R$.
   (d) R is the relation on the set of positive real numbers given by xRy if and only if x/y is a rational number.
      reflexive: Yes. For every positive real number x, $x/x = 1$ which is rational. So $(x, x) \in R$ for every $x \in \mathbb{R}^+$.
      irreflexive: No. See previous.
      symmetric: Yes. Since the reciprocal of a rational number is rational we have $x/y \in \mathbb{Q}$ implies $y/x \in \mathbb{Q}$. Thus $(x, y) \in R$ implies $(y, x) \in R$.

antisymmetric: No. Consider for example, $2/3, 3/2 \in \mathbb{Q}$ but $2 \neq 3$.

transitive: Yes. Assume $(x, y), (y, z) \in R$. Then $x/y, y/z \in \mathbb{Q}$. Now, $x/z = (x/y) \cdot (y/z)$. Since the product of 2 rational numbers is rational (see text p. 75 problem 26), we have $x/z \in \mathbb{Q}$. Thus $(x, z) \in R$.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \lfloor x \rfloor$. Define the relation R on the set of real numbers by $R = graph(f)$.

reflexive: No. Consider, for example, $1.5 \in \mathbb{R}$, but $(1.5, 1.5) \notin R$ since $f(1.5)$ equals 1 not 1.5.

irreflexive: No. $f(1) = 1$, so $(1, 1) \in R$.

symmetric: No. $f(1.5) = 1$ but $f(1) = 1 \neq 1.5$, so $(1.5, 1) \in R$, but $(1, 1.5) \notin R$.

antisymmetric: Yes. Notice that if $(x, y), (y, x) \in R$, then both x and y are in the range. The range of this function is the set of integers, so both x and y are integers. But the floor function maps integers to themselves, so $y = f(x) = x$.

transitive: Yes. Suppose $(x, y), (y, z) \in R$. Then $f(x) = y$ and $f(y) = z$. Notice this means y is in the range and so is an integer. Thus $z = f(y) = y$. Now this tells us $(x, z) = (x, y)$ which we know is in R.

(f) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \lfloor x \rfloor$. Define the relation R on the set of real numbers by aRb iff $f(a) = f(b)$.

reflexive: Yes. For any real number, x, we have $f(x) = f(x)$, so $(x, x) \in R$.

irreflexive: No. See previous.

symmetric: Yes. For any pair of real numbers, x and y, if $(x, y) \in R$ then $f(x) = f(y)$. But equality is symmetric, so $f(y) = f(x)$ and so $(y, x) \in R$.

antisymmetric: No. Take, for example, $(1.5, 1), (1, 1.5) \in R$ but $1.5 \neq 1$.

transitive: Yes. For any real numbers, x, y, and z, if $(x, y), (y, z) \in R$ then $f(x) = f(y)$ and $f(y) = f(z)$. This implies $f(x) = f(z)$ by substitution, so $(x, z) \in R$.

2. Let R be the relation $R = \{(a, c), (b, b), (b, c), (c, a)\}$ and S the relation $S = \{(a, a), (a, b), (b, c), (c, a)\}$ is a relation on $A = \{a, b, c\}$.
   (a) $R^2 = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$.
   (b) $S \circ R = \{(a, a), (b, a), (b, c), (c, a), (c, b)\}$.
   (c) $R^{-1} = \{(c, a), (b, b), (c, b), (a, c)\}$
3. The matrix below is the matrix for a relation.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

   (a) The relation does not satisfy any of the properties.

(b)

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(f)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

4. Suppose R is a relation on A. Using composition is associative and mathematical induction, prove that $R^n \circ R = R \circ R^n$.

   Basis: $R^1 \circ R = R \circ R = R \circ R^1$ is clear.

   Induction Step: Let $k \in \mathbb{Z}^+$ and assume $R^k \circ R = R \circ R^k$. Prove $R^{k+1} \circ R = R \circ R^{k+1}$.

   $R^{k+1} \circ R = (R^k \circ R) \circ R$ by the definition of R^{k+1}
   $= (R \circ R^k) \circ R$ by the induction hypothesis
   $= R \circ (R^k \circ R)$ since composition is associative
   $= R \circ R^{k+1}$ by the definition of R^{k+1}.
5. This is Theorem 1 on page 479 in the text.
6. Let A be a set and let R and S be relations on A. If R and S satisfy the property given, does the relation given have to satisfy the same property? Prove or disprove each answer.
   (a) $R \cup S$ is reflexive: Let $a \in A$. Since R is reflexive, $(a, a) \in R$. Thus $(a, a) \in R \cup S$ which shows $R \cup S$ is reflexive.

(b) R^{-1} is reflexive. Let $a \in A$. Since R is reflexive, $(a, a) \in R$. Thus (a, a) is also in R since reversing the order of the elements in this ordered pair gives you the same ordered pair back.

(c) Let $(x, y) \in t(R)$. Recall $t(R) = \cup_{k=1}^{\infty} R^k$, so $(x, y) \in R^k$ for some integer k. We claim R^k is symmetric for all positive integers k. If this is true, then we have $(y, x) \in R^k \subseteq t(R)$ which proves $t(R)$ is symmetric.
Now we show R^k is symmetric if R is by induction on k.

Basis: $R^1 = R$. Since R is symmetric, this is clear.

Induction Step: Let k be a positive integer and assume R^k is symmetric. Prove R^{k+1} is symmetric. Let $(a, b) \in R^{k+1} = R \circ R^k$. Then there is a $c \in A$ such that $(a, c) \in R^k$ and $(c, b) \in R$. Since R and R^k are symmetric, $(c, a) \in R^k$ and $(b, c) \in R$. Thus $(b, a) \in R^k \circ R = R^{k+1}$. Therefore R^{k+1} is symmetric.

(d) $R \circ S$ is not necessarily symmetric: Consider the case where $A = \{a, b, c\}$, $R = \{(a, b), (b, a)\}$, and $S = \{(b, c), (c, b)\}$. Then $(c, a) \in R \circ S$, but $(a, c) \notin R \circ S$.

(e) $R \oplus S$ is not necessarily antisymmetric: Let $A = \{a, b\}$. The relations $R = \{(a, b)\}$ and $S = \{(b, a)\}$ are antisymmetric, but $R \oplus S = \{(a, b), (b, a)\}$ is not antisymmetric. (to *disprove* the property must hold true, a counter example is sufficient. An example does *not* prove a property must hold for all possible relations.)

(f) R^n for any positive integer n is not necessarily antisymmetric: Let $A = \{a, b, c, d\}$ and let $R = \{(a, b), (b, c), (c, d), (d, a)\}$. Then R is antisymmetric (vacuously). $R^2 = \{(a, c), (c, a), (b, d), (d, b)\}$ which is not antisymmetric.

(g) $r(R)$ is transitive: Let $(x, y), (y, z) \in r(R) = \Delta \cup R$. Then $x = y$ or $y = z$ or $(x, y), (y, z) \in R$. If $x = y$ then $(x, z) = (y, z) \in r(R)$. If $y = z$ then $(x, y) = (x, z) \in r(R)$. If $(x, y), (y, z) \in R$ then $(x, z) \in R \subseteq r(R)$ since R is transitive. This shows $r(R)$ is transitive.

(h) R^{-1} is transitive: Let $a, b, c \in A$ be such that $(a, b), (b, c) \in R^{-1}$. Then $(b, a), (c, b) \in R$. Since R is transitive, $(c, a) \in R$. Thus $(a, c) \in R^{-1}$ which shows R^{-1} is transitive if R is.

7. *Proof.* $(x, y) \in (R_2 \circ R_1)^{-1}$
$\Leftrightarrow (y, x) \in (R_2 \circ R_1)$
$\Leftrightarrow \exists t \in T (y, t) \in R_1$ and $(t, x) \in R_2$
$\Leftrightarrow \exists t \in T (t, y) \in R_1^{-1}$ and $(x, t) \in R_2^{-1}$
$\Leftrightarrow (x, y) \in (R_1^{-1} \circ R_2^{-1})$
This shows $(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$. □

8. Let $A = \{1, 2, 3, 4, 5\}$ and let R_1 be the relation give by $(n, m) \in R_1$ iff $n \equiv m (\text{mod } 3)$.

(a)

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(b) Draw 5 vertices and label them 1, 2, 3, 4, 5. Add a directed loop to each vertex. Add a directed edge from 1 to 4 and from 4 to 1. Add a directed edge from 2 to 5 and from 5 to 2.

9. True or false, prove or disprove: If R and S are relations on A, then
   (a) True: $(R \cap S) \cup \Delta = (R \cup \Delta) \cap (S \cup \Delta)$, where $\Delta = \{(a,a)|a \in A\}$.
   (b) False: $\Delta \not\subseteq r(R) - r(S)$ since $\Delta \subseteq r(S)$
   (c) True: $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$, so $(R \cup S) \cup (R \cup S)^{-1} = (R \cup R^{-1}) \cup (S \cup S^{-1})$.
   (d) False: Consider $A = \{a, b\}$ and $R = \{(a, b)\}$. Then $s(R) = \{(a, b), (b, a)\}$, $s(R)^2 = \{(a, a), (b, b)\}$, $R^2 = \emptyset$, and $s(R^2) = \emptyset$.
   (e) True: To prove we need to show $\cup_{n=1}^{\infty}(R^{-1})^n = (\cup_{n=1}^{\infty} R^n)^{-1}$. We will use the property: $(R^{-1})^n = (R^n)^{-1}$ for any positive integer n, which we prove below.

$(x, y) \in \cup_{n=1}^{\infty}(R^{-1})^n$
$\Leftrightarrow (x, y) \in (R^{-1})^n$ for some positive integer n
$\Leftrightarrow (x, y) \in (R^n)^{-1}$ for some positive integer n
$\Leftrightarrow (y, x) \in (R^n)$ for some positive integer n
$\Leftrightarrow (y, x) \in \cup_{n=1}^{\infty}(R^n)$
$\Leftrightarrow (x, y) \in (\cup_{n=1}^{\infty}(R^n))^{-1}$

Proof that $(R^{-1})^n = (R^n)^{-1}$ for any positive integer n, by induction on n:

Basis: Clearly $(R^{-1})^1 = (R^1)^{-1}$, since $S^1 = S$ for any relation S.

Induction Hypothesis: Let n be a positive integer and assume $(R^{-1})^n = (R^n)^{-1}$.

Induction Step: Prove $(R^{-1})^{n+1} = (R^{n+1})^{-1}$

$(x, y) \in (R^{-1})^{n+1}$
$\Leftrightarrow \exists c \in A$ with $(x, c) \in (R^{-1})^n$ and $(c, y) \in R^{-1}$
$\Leftrightarrow \exists c \in A$ with $(x, c) \in (R^n)^{-1}$ and $(c, y) \in R^{-1}$, by the induction hypothesis
$\Leftrightarrow \exists c \in A$ with $(c, x) \in (R^n)$ and $(y, c) \in R$
$\Leftrightarrow (y, x) \in (R^{n+1})$
$\Leftrightarrow (x, y) \in (R^{n+1})^{-1}$

   (f) False: $\cup_{n=1}^{\infty}(R \circ S)^n$ is not necessarily equal to $(\cup_{n=1}^{\infty} R^n) \circ (\cup_{n=1}^{\infty} S^n)$. This is easier to see if you write out the union using ellipses.

10. Let R be the relation on the set of all integers given by nRm if and only if $nm < 0$.
   (a) $(m, n) \in r(R)$ if and only if $nm < 0$ or $m = n$.
   (b) $(m, n) \in s(R)$ if and only if $nm < 0$.
   (c) $(m, n) \in t(R)$ if and only if $m \neq 0$ and $n \neq 0$. (there are other equivalent ways to define this set)

11. Let A be a set and let R and S be relations on A. If R and S satisfy the property given, does the relation given have to satisfy the same property? Prove or disprove each answer.
    (a) $R - S$ is not an equivalence relation: Since both R and S are reflexive, they both contain $\Delta = \{(a, a)|a \in A\}$, so $R - S$ does not contain Δ and so is not reflexive.
    (b) R^n for any positive integer n is an equivalence relation: Proof by induction on n.
        Basis: R^1 is an equivalence relation by our original assumption.
        Induction Hypothesis: Let n be a positive integer and assume R^n is an equivalence relation.
        Induction Step: Prove R^{n+1} is an equivalence relation.
            Reflexive: Let $a \in A$. Since R and R^n are reflexive, $(a, a) \in R$ and $(a, a) \in R^n$. Thus $(a, a) \in R^{n+1} = R^n \circ R$.
            Symmetric: Let $a, b \in A$ be such that $(a, b) \in R^{n+1} = R^n \circ R$. Then there is a $t \in A$ such that $(a, t) \in R$ and $(t, b) \in R^n$. Now, since both R and R^n are symmetric, $(t, a) \in R$ and $(b, t) \in R^n$. Thus $(b, a) \in R \circ R^n = R^{n+1}$.
            Transitive: Let $a, b, c \in A$ be such that $(a, b), (b, c) \in R^{n+1} = R^n \circ R$. Then there are $s, t \in A$ such that $(a, s), (b, t) \in R$ and $(s, b), (t, c) \in R^n$. Since R is transitive, $R^n \subseteq R$, so $(s, b) \in R$. Since R is transitive and $(a, s), (s, b), (b, t) \in R$, we get $(a, t) \in R$. Thus since $(a, t) \in R$ and $(t, c) \in R^n$ we find $(a, c) \in R^n \circ R = R^{n+1}$.
    (c) $R \circ S$ is not necessarily a partial order. This is not necessarily antisymmetric, nor transitive. Counterexample: Define R on the set of integers by $\leq$ and define S on the set of integers by $\geq$. Then $R \circ S$ is not antisymmetric.
    (d) $R \oplus S$ is not a Partial Order: Since both R and S are partial orders, $\Delta = \{(a, a)|a \in A\}$, is contained in both R and S. Thus $R \oplus S$ does not contain Δ, and so $R \oplus S$ cannot be reflexive.
12. Which of the following relations are equivalence relations, which are partial orders, and which are neither? For the relations that are equivalence relations find the equivalence classes. For the ones that are neither equivalence relations nor partial orders name the property(ies) that fails.
    (a) The relation R on the set of Computer Science majors at FSU where aRb iff a and b are currently enrolled in the same course.
        This relation is not necessarily transitive, so not an equivalence relation nor a partial order.
    (b) The relation R on the set of integers where $(m, n) \in R$ if and only if $mn \equiv 2 (\text{mod } 2)$.
        Neither: Not reflexive nor transitive.
    (c) The relation R on the set of ordered pairs of integers where $(a, b)R(c, d)$ iff $a = c$ or $b = d$
        Neither: Not transitive

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \lceil x \rceil$. Define the relation R on $\mathbb{R}$ by $(x, y) \in R$ if and only if $f(x) = f(y)$.
Equivalence Relation: Equivalence classes are of the form $(n, n+1]$ where $n \in \mathbb{Z}$

(e) The relation R on the set of all subsets of $\{1, 2, 3, 4\}$ where SRT means $S \subseteq T$
Partial Order

(f) The relation R on the set of positive integers where $(m, n) \in R$ if and only if $gcd(m, n) = max\{m, n\}$.
Equivalence Relation. The equvalence classes are of the form $[m] = \{m\}$, since the only integer related to m using this relation is m.

(g) Let $G = (V, E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that $u = v$.
Equivalence Relation. The equivalence class of a vertex v consists of all the vertices (including v) in the same component as v.

(h) The relation R on the set of ordered pairs $\mathbb{Z}^+ \times \mathbb{Z}^+$ of positive integers defined by

$$(a, b)R(c, d) \Leftrightarrow a + d = b + c.$$

Equivalence Relation. $[(a, b)] = \{(c, d) | a - b = c - d\}$

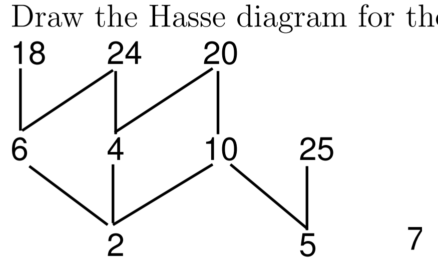
13. Let R be the relation on the set of ordered pairs of positive integers such that $(a, b)R(c, d)$ if and only if $ad = bc$.
(a) Prove R is an equivalence relation.

Proof. **Reflexive:** Let $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Since $ab = ba$, we have $(a, b)R(a, b)$.
Symmetric: Let $(a, b)R(c, d)$. Then $ad = bc$. But this implies $cb = da$ so $(c, d)R(a, b)$.
Transitive: Let $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then $ad = bc$ and $cf = de$. If we take the first equation and multiply through by f we get $adf = bcf$. Then using the second equation we substitute for cf to get $adf = bde$. Thus $d(af - be) = 0$. Since the integers are all positive we know $d \neq 0$ and so $af = be$. Therefore $(a, b)R(e, f)$. □

(b) Find the equivalence class of (a, b) where $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$: $[(a, b)] = \{(c, d) | ad = bc\} = \{(c, d) |$ the rational numbers a/b and c/d are equal $\}$.

14. This is example 3.2.1 in *Equivalence Relations*

15. Suppose $A = \{2, 4, 5, 6, 7, 10, 18, 20, 24, 25\}$ and R is the partial order relation $(x, y) \in R$ iff $x | y$.
(a) Draw the Hasse diagram for the relation.



(b) Find all minimal elements.
2, 5, and 7

(c) Find all maximal elements.
7, 18, 24, 20, and 25

(d) Find all upper bounds for $\{6\}$.
6, 18, and 24

(e) Find all lower bounds for $\{6\}$.
6 and 2

(f) Find the least upper bound for $\{6\}$.
6

(g) Find the greatest lower bound for $\{6\}$.
6

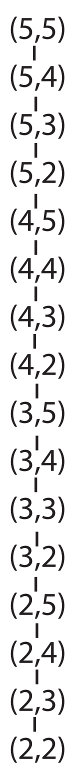
(h) Find the least element.
none

(i) Find the greatest element.
none

(j) Is this a lattice?
no

16. Suppose $A = \{2, 3, 4, 5\}$ has the usual "less than or equal" order on integers. Find each of the following for the case where R is the lexicographic partial order relation on $A \times A$ and where R is the product partial order relation on $A \times A$.

Lexicographic Order: (a) Draw the Hasse diagram for the relation.



(b) Find all minimal elements.

$(2,2)$

(c) Find all maximal elements.
$(5,5)$

(d) Find all upper bounds for $\{(2,3),(3,2)\}$.
$(3,2),(3,3),(3,4),(3,5),(4,2),(4,3),(4,4),(4,5),(5,2),(5,3),(5,4),(5,5)$

(e) Find all lower bounds for $\{(2,3),(3,2)\}$.
$(2,3),(2,2)$

(f) Find the least upper bound for $\{(2,3),(3,2)\}$.
$(3,2)$

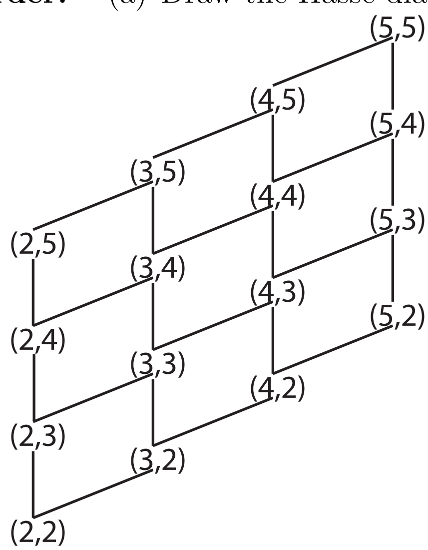
(g) Find the greatest lower bound for $\{(2,3),(3,2)\}$.
$(2,3)$

(h) Find the least element.
$(2,2)$

(i) Find the greatest element.
$(5,5)$

(j) Is this a lattice?
Yes.

Product Order: (a) Draw the Hasse diagram for the relation.



(b) Find all minimal elements.
$(2,2)$

(c) Find all maximal elements.
$(5,5)$

(d) Find all upper bounds for $\{(2,3),(3,2)\}$.
$(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)$

(e) Find all lower bounds for $\{(2,3),(3,2)\}$.
$(2,2)$

(f) Find the least upper bound for $\{(2,3),(3,2)\}$.
$(3,3)$

(g) Find the greatest lower bound for $\{(2,3),(3,2)\}$.
$(2,2)$

(h) Find the least element.
$(2,2)$

(i) Find the greatest element.
$(5,5)$

(j) Is this a lattice?
Yes

17. Carefully prove the following relations are partial orders.

(a) Recall the product order: Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be posets. Define the relation $\preceq$ on $A_1 \times A_2$ by $(a_1, a_2) \preceq (b_1, b_2)$ if and only if $a_1 \preceq_1 b_1$ and $a_2 \preceq_2 b_2$. Prove the product order is a partial order.

Proof. **Reflexive:** Let $(a, b) \in A_1 \times A_2$. Since $\preceq_1$ and $\preceq_2$ are partial order, $a \preceq_1 a$ and $b \preceq_2 b$. Therefore $(a, b) \preceq (a, b)$.

Antisymmetric: Let $(a_1, a_2), (b_1, b_2) \in A_1 \times A_2$ and assume $(a_1, a_2) \preceq (b_1, b_2)$ and $(b_1, b_2) \preceq (a_1, a_2)$. Then $a_1 \preceq_1 b_1$, $a_2 \preceq_2 b_2$, $b_1 \preceq_2 a_1$, and $b_2 \preceq_2 a_2$. Since both $\preceq_1$ and $\preceq_2$ are partial orders, $a_1 = b_1$ and $a_2 = b_2$. This shows $(a_1, a_2) = (b_1, b_2)$.

Transitive: $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in A_1 \times A_2$ and assume $(a_1, a_2) \preceq (b_1, b_2)$ and $(b_1, b_2) \preceq (c_1, c_2)$. Then $a_1 \preceq_1 b_1$, $b_1 \preceq_1 c_1$, $a_2 \preceq_2 b_2$ and $b_2 \preceq_2 c_2$. Since both $\preceq_1$ and $\preceq_2$ are partial orders, $a_1 \preceq_1 c_1$ and $a_2 \preceq_2 c_2$. This shows $(a_1, a_2) \preceq (c_1, c_2)$.

□

(b) Let $(B, \preceq_B)$ be a poset and let A be a set. Recall the set $FUN(A, B)$ defined to be the set of all functions with domain A and codomain B. Prove the relation $\preceq_F$ on $FUN(A, B)$ defined by $f \preceq_F g$ iff $f(t) \preceq_B g(t)$ $\forall t \in A$ is a partial order.

reflexive: Let $f \in FUN(A, B)$. Since $\preceq_B$ is a partial order $f(x) \preceq_B f(x)$ for all $x \in A$. Thus $f \preceq_F f$.

antisymmetric: Let $f, g \in FUN(A, B)$ be such that $f \preceq_F g$ and $g \preceq_F f$. Then
$\forall x \in A(f(x) \preceq_B g(x))$ and $\forall x \in A(g(x) \preceq_B f(x))$ by the definition of $\preceq_F$
$\Rightarrow \forall x \in A[(f(x) \preceq_B g(x))$ and $(g(x) \preceq_B f(x))]$
$\Rightarrow \forall x \in Af(x) = g(x)$ since $\preceq_B$ is a partial order
$\Rightarrow f = g$ by the definition of function equality

transitive: Let $f, g, h \in FUN(A, B)$ be such that $f \preceq_F g$ and $g \preceq_F h$.
$\Rightarrow \forall x \in A(f(x) \preceq_B g(x))$ and $\forall x \in A(g(x) \preceq_B h(x))$ by the definition of $\preceq_F$
$\Rightarrow \forall x \in A[(f(x) \preceq_B g(x))$ and $(g(x) \preceq_B h(x))]$
$\Rightarrow \forall x \in A[f(x) \preceq_B h(x)]$ since $\preceq_B$ is a partial order
$\Rightarrow f \preceq_F h$ by the definition of $\preceq_F$.

(c) **reflexive:** Let $(a, b) \in A \times B$. Since $\preceq_A$ is a partial order $a \preceq_A a$. Thus $(a, b) \preceq_L (a, b)$.

antisymmetric: Suppose $(a, b) \preceq_L (c, d)$ and $(c, d) \preceq_L (a, b)$.
Then $a \preceq_A c$ and $c \preceq_A a$ by the definition of $\preceq_L$. Since $\preceq_A$ is a partial order this gives us $a = c$. Then $b \preceq_B d$ and $d \preceq_B b$ by the definition of $\preceq_L$. Thus $b = d$ since $\preceq_B$ is a partial order. Thus we now have $(a, b) = (c, d)$.

transitive: Let $(a, b) \preceq_L (c, d)$ and $(c, d) \preceq_L (e, f)$. Then $a \preceq_A c$ and $c \preceq_A e$. Since $\preceq_A$ is a partial order we get $a \preceq_A e$.
If $a \neq e$. Then $(a, b) \preceq_L (e, f)$ by the definition of $\preceq_L$.
Suppose $a = e$. Then c must equal a and e as well and so $b \preceq_B d$ and $d \preceq_B f$. Since $\preceq_B$ is a partial order we have $b \preceq_B f$.
This shows $(a, b) \preceq_L (e, f)$.

18. Suppose $(A, \preceq)$ is a poset such that every nonempty subset of A has a least element. Let $x, y \in A$. Then $\{x, y\}$ has a least element which must be x or y. Thus $x \preceq y$ or $y \preceq x$. This shows A is a total order.
19. This is Theorem 4.19.1 in *Partial Orderings*.